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MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Ordinary Level

CANDIDATE NAME	

CENTRE NUMBER	S					INDEX NUMBER					

ADDITIONAL MATHEMATICS

4049/01

Paper 1

October/November 2023

2 hours 15 minutes

Candidates answer on the Question Paper.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

This document consists of **19** printed pages and **1** blank page.

Singapore Examinations and Assessment Board

Cambridge Assessment International Education

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!} = \dfrac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}bc\sin A$$

1 Find the value of the constant c such that the line $y=2x+c$ is a tangent to the curve $y=x^2+3x+1$. [3]

Step 1: Let $x^2+3x+1=2x+c$

$x^2+x+(1-c)=0$

Step 2: $b^2-4ac=0$

$1-4(1)(1-c)=0$

$1-4+4c=0$

$4c=3$

$c=\frac{3}{4}$

2 Express $\frac{18+11x-2x^2}{(x-1)(x+2)^2}$ in partial fractions. [5]

$$\frac{18+11x-2x^2}{(x-1)(x+2)^2}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$$

By Cover Up Rule: Let $x=1$

$$A=\frac{18+11-2}{3^2}=3$$

$$\frac{18+11x-2x^2}{(x-1)(x+2)^2}=\frac{3}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$$

$$=\frac{3(x+2)^2+B(x-1)(x+2)+C(x-1)}{(x-1)(x+2)^2}$$

By Substitution:

Let $x=-2$: $18-22-8=C(-3)$

$-12=-3C$

$\therefore C=4$

Let $x=0$: $18=3(4)+B(-1)(2)+4(-1)$

$18=12-2B-4$

$2B=-10$

$B=-5$

$$\therefore \frac{18+11x-2x^2}{(x-1)(x+2)^2}=\frac{3}{(x-1)}-\frac{5}{(x+2)}+\frac{4}{(x+2)^2}$$

3 Prove that $\dfrac{\cos^2\theta}{(\operatorname{cosec}\theta-1)(\operatorname{cosec}\theta+1)}+\dfrac{\sin^2\theta}{(\sec\theta-1)(\sec\theta+1)}=1.$ [5]

$$LHS = \frac{\cos^2\theta}{(\operatorname{cosec}\theta-1)(\operatorname{cosec}\theta+1)}+\frac{\sin^2\theta}{(\sec\theta-1)(\sec\theta+1)}$$

$$= \frac{\cos^2\theta}{\operatorname{cosec}^2\theta - 1} + \frac{\sin^2\theta}{\sec^2\theta - 1}$$

$$= \frac{\cos^2\theta}{\cot^2\theta} + \frac{\sin^2\theta}{\tan^2\theta}$$

$$= \frac{\cos^2\theta\tan^2\theta + \cot^2\theta\sin^2\theta}{(\cot^2\theta)(\tan^2\theta)}$$

$$= \frac{\cancel{\cos^2\theta}\left(\frac{\sin^2\theta}{\cancel{\cos^2\theta}}\right) + \left(\frac{\cos^2\theta}{\cancel{\sin^2\theta}}\right)\cancel{\sin^2\theta}}{1}$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

$$= RHS \text{ (proven)}$$

Recap Identities:
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

- $\cot\theta = \dfrac{1}{\tan\theta}$

$\therefore \cot^2\theta = \dfrac{1}{\tan^2\theta}$

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The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

4 **(a)** Find $\frac{d}{dx}(xe^{-2x})$ [3]

$= x(-2e^{-2x}) + e^{-2x}(1)$

$= \underline{-2xe^{-2x} + e^{-2x}}$

(b) Hence find $\int xe^{-2x}\,dx$. [4]

from (a): $\int -2xe^{-2x} + e^{-2x}\,dx = xe^{-2x} + C$

$-2\int xe^{-2x}\,dx + \frac{e^{-2x}}{-2} = xe^{-2x} + C$

$\therefore -2\int xe^{-2x}\,dx = \frac{1}{2}e^{-2x} + xe^{-2x} + C$

$\int xe^{-2x}\,dx = \frac{-1}{4}e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{2}C$

$= \underline{\frac{-1}{4}e^{-2x}(1+2x) + B}$

(where $B = -\frac{1}{2}C$)

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

5 Solve the equation $\dfrac{\cos\theta+4\sin\theta}{2\cos\theta+\sin\theta}=\cot\theta$ for $-\dfrac{\pi}{2}\leqslant\theta\leqslant\dfrac{\pi}{2}$. [6]

$$\frac{\cos\theta+4\sin\theta}{2\cos\theta+\sin\theta}=\frac{\cos\theta}{\sin\theta}$$

$$\cancel{\sin\theta\cos\theta}+4\sin^2\theta=2\cos^2\theta+\cancel{\sin\theta\cos\theta}$$

$$4\sin^2\theta-2\cos^2\theta=0$$

$$2\sin^2\theta-\cos^2\theta=0$$

$$2\sin^2\theta-(1-\sin^2\theta)=0$$

$$3\sin^2\theta=1$$

$$\sin^2\theta=\frac{1}{3}$$

$$\sin\theta=\pm\frac{1}{\sqrt{3}}$$

Step 1: Basic angle for $\theta = 0.61548$ rad.

Step 2: $-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$

$\therefore$ $\theta = -0.615$ or 0.615

6 The function f is given by $f(x) = \dfrac{ax^2}{x-a}$, for $x > a$, where a is a positive constant.

(a) Find $f'(x)$. [2]

$$f'(x) = \frac{(x-a)(2ax) - (ax^2)(1)}{(x-a)^2}$$

$$= \frac{2ax^2 - 2a^2x - ax^2}{(x-a)^2}$$

$$= \frac{ax^2 - 2a^2x}{(x-a)^2}$$

$$= \frac{ax(x-2a)}{(x-a)^2}$$

The function g, defined for $x > a$, has the property that $g'(x) = (x-a)^2 f'(x)$.
g decreases for $a < x < 8$.

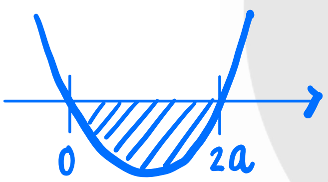
(b) Find the value of a. [4]

Step 1:

$$g'(x) = \cancel{(x-a)^2} \cdot \frac{ax(x-2a)}{\cancel{(x-a)^2}}$$

$$= ax(x-2a)$$

Step 2: It is given that $g'(x) < 0$ for $a < x < 8$.
and a is a positive constant



Let $ax(x-2a) < 0$

$\therefore x(x-2a) < 0$

$0 < x < 2a$

Step 3: Since $x > a$ and $0 < x < 2a$,
hence $a < x < 2a$.

Comparing to $a < x < 8$, $2a = 8$

$\therefore a = 4$

7 Find the set of values of the constant k for which the curve $y = kx^2 + 4x + k - 3$ lies completely below the x-axis. [6]

Step 1 : For a curve to lie completely below the x-axis, $k < 0$.

Step 2 : When a curve lies completely below the x-axis, there are No Real Roots.

$b^2 - 4ac < 0$

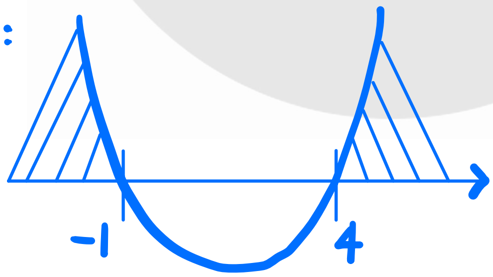
$4^2 - 4k(k-3) < 0$

$16 - 4k^2 + 12k < 0$

(Divide both sides by -4) $\therefore k^2 - 3k - 4 > 0$

$(k-4)(k+1) > 0$

Step 3 :



From the graph, $k < -1$ or $k > 4$.
However, from step 1, $k < 0$.
Hence, $k < -1$

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8 The line $y-2x=12$ intersects the curve $x^2-xy+y^2=63$ at two points.
Find the coordinates of these two points. [5]

$y = 2x + 12$ —— (1)
$x^2 - xy + y^2 = 63$ —— (2)

Sub (1) into (2) :

$x^2 - x(2x+12) + (2x+12)^2 = 63$

$x^2 - 2x^2 - 12x + 4x^2 + 48x + 144 - 63 = 0$

$3x^2 + 36x + 81 = 0$

$x^2 + 12x + 27 = 0$

$(x+9)(x+3) = 0$

$x = -3$ or $x = -9$

When $x = -3$, $y = 2(-3) + 12$
$= 6$

When $x = -9$, $y = 2(-9) + 12$
$= -6$

The coordinates of the intersection points are (-3, 6) and (-9, -6)

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9 A curve has equation $y = 1 + x - \frac{x^2}{2} - 2x^3$.

(a) Find the x-coordinates of the stationary points, A and B, on the curve. [4]

$y = 1 + x - \frac{1}{2}x^2 - 2x^3$

$\frac{dy}{dx} = 1 - x - 6x^2$

Let $\frac{dy}{dx} = 0$: $-6x^2 - x + 1 = 0$

$6x^2 + x - 1 = 0$

$(3x - 1)(2x + 1) = 0$

$\therefore$ $x = -\frac{1}{2}$ or $x = \frac{1}{3}$

It is given that P is a point on the curve where the **gradient is a maximum**.

(b) Show that P and the midpoint of AB have the same x-coordinate. [4]

Let gradient be G :

Step 1: $G = 1 - x - 6x^2$

$$\frac{dG}{dx} = -1 - 12x$$

Step 2: Let $\frac{dG}{dx} = 0$

$$\therefore -1 - 12x = 0$$
$$12x = -1$$
$$x = \frac{-1}{12}$$

Step 3: $\frac{d^2G}{dx^2} = -12 < 0$ $\boxed{\text{2nd Derivative Test}}$

so when $x = \frac{-1}{12}$, gradient is maximum.

i.e. x-coordinate of $P = \frac{-1}{12}$

x-coordinate of midpoint of $AB = \dfrac{-\frac{1}{2} + \frac{1}{3}}{2}$

$= \frac{-1}{12}$ (shown)

10 A circle, with centre C, has equation $x^2 + y^2 + 10x - 24y = 0$.

(a) Find the coordinates of C and the radius of the circle. [4]

$$x^2 + 10x + y^2 - 24y = 0$$

$$(x+5)^2 - 5^2 + (y-12)^2 - 12^2 = 0$$

$$(x+5)^2 + (y-12)^2 = 13^2$$

$\therefore$ $C = (-5, 12)$ and radius $r = 13$

(b) Find the coordinates of the points at which the circle intersects the y-axis. [2]

Let $x = 0$: $y^2 - 24y = 0$

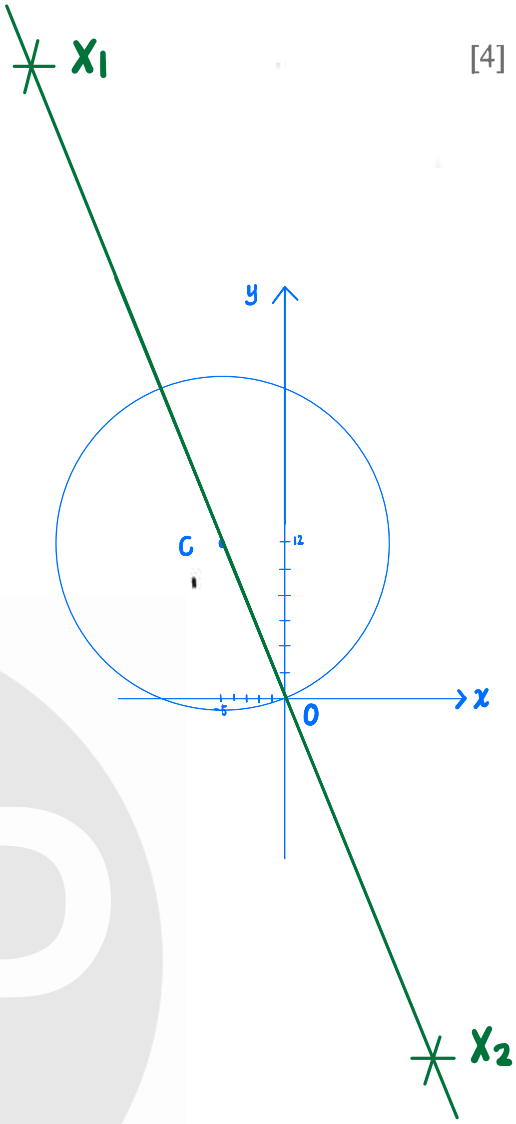
$$y(y-24) = 0$$

$$y = 0 \quad \text{or} \quad y = 24$$

$\therefore$ coordinates are $(0,0)$ and $(0,24)$.

The point X is on the line which passes through C and the origin, O. It is given that the distance CX is three times the distance OC.

(c) Find the coordinates of the possible positions of X. [4]



Using Ratio Theorem:

$X_1(a, b)$

3

$C(-5, 12)$

1

$(0, 0)$

$$\frac{3(0)+1(a)}{4} = -5 \qquad \frac{3(0)+1(b)}{4} = 12$$

$\therefore\ a = -20 \qquad \therefore\ b = 48$

Hence, $X_1 = (-20, 48)$

$C(-5, 12)$

1

$(0, 0)$

2

$X_2(c, d)$

$$\frac{1(c)+2(-5)}{3} = 0 \qquad \frac{1(d)+2(12)}{3} = 0$$

$\therefore\ c = 10 \qquad \therefore\ d = -24$

Hence, $X_2 = (10, -24)$

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11 A circular patch of water of negligible thickness is expanding on a thin piece of paper. At time t seconds the rate of change of the radius, r cm, of the patch is given by

$$\frac{dr}{dt} = \frac{k}{2t+1} \text{ cm/s, where } k \text{ is a constant.}$$

Initially the radius of the patch is 1 cm and the radius is increasing at a rate of 0.5 cm/s.

(a) Show that $k = 0.5$. [1]

Let $0.5 = \frac{k}{2(0)+1}$

$\therefore k = 0.5$

(b) Find an expression for r in terms of t. [4]

$$r = \int \frac{0.5}{2t+1}\,dt$$
$$= 0.5\int \frac{1}{2t+1}\,dt$$
$$= 0.5\left[\frac{\ln(2t+1)}{2}\right] + C$$
$$= \frac{1}{4}\ln(2t+1) + C$$

Since $r = 1$ cm when $t = 0$,

$\therefore 1 = \frac{1}{4}\ln 1 + C$

$C = 1$

Hence, $r = \frac{1}{4}\ln(2t+1) + 1$

(c) Hence find the rate of increase of the area of the patch after the patch has been expanding for 3 seconds. [4]

Step 1: $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

Step 2: When $t = 3s$, $r = \frac{1}{4}\ln(6+1) + 1$

$= 1.4865$ cm.

Step 3: $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 2\pi(1.4865) \times \frac{0.5}{2(3)+1}$

$= 0.66713$

$= \underline{0.667 \text{ cm}^2/s}$

12 A ball is thrown vertically upwards. Its height, h m, above the ground at time t seconds after being thrown is given by the formula $h = 1.75 + 5t - 5t^2$.

(a) State the height above the ground from which the ball is thrown. [1]

When $t = 0$ s, $\underline{h = 1.75 \text{ m}}$

(b) Express h in the form $a + b(t+c)^2$ where a, b and c are constants to be determined. [3]

$$h = -5(t^2 - t) + 1.75$$
$$= -5\left[\left(t - \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2\right] + 1.75$$
$$= -5\left(t - \tfrac{1}{2}\right)^2 + 1.25 + 1.75$$
$$= \underline{3 - 5\left(t - \tfrac{1}{2}\right)^2}$$

where $a = 3$, $b = -5$ and $c = -0.5$

(c) Hence state the maximum height attained by the ball and the time at which this occurs. [2]

Max. height = $\underline{3 \text{ m}}$

when $t = \underline{0.5 \text{ s}}$

(d) The ball hits the ground. Explain why the time taken for the ball to hit the ground is **not** twice the time found in **part (c)**. [1]

Because the ball wasn't thrown up from the ground level. It was thrown from an initial height of 1.75 m.

(e) Find the length of time for which the ball is at least 2 m above the ground. [3]

Step 1: Let $h = 2$

$$\therefore 3 - 5\left(t - \tfrac{1}{2}\right)^2 = 2$$

$$5\left(t - \tfrac{1}{2}\right)^2 = 1$$

$$t - \tfrac{1}{2} = \pm\sqrt{\tfrac{1}{5}}$$

$$t = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{5}}$$

$$= 0.052786\text{ s} \text{ or } 0.94721\text{ s}$$

Step 2:

$$\text{duration} = 0.94721 - 0.052786$$

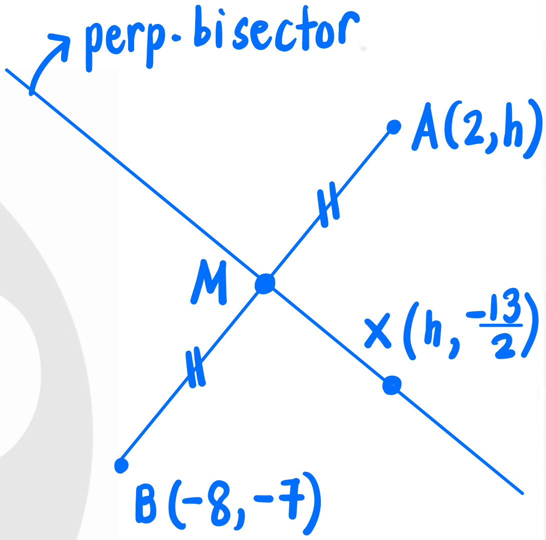
$$= 0.894\text{ s}$$

13 The perpendicular bisector of the line joining the points $A(2, h)$ and $B(-8, -7)$ passes through the point $X\left(h, -\frac{13}{2}\right)$, where h is a constant.

(a) Express the gradient of the perpendicular bisector of the line AB in terms of h. [3]

Step 1: gradient $AB = \frac{h-(-7)}{2-(-8)}$

$= \frac{h+7}{10}$



Step 2: gradient of perp. bisector $= \frac{-10}{h+7}$

$\boxed{\text{for perpendicular lines, } m_1 \times m_2 = -1}$

(b) Hence find the possible values of h. [7]

Step 1:

midpoint of $AB = \left(\frac{-8+2}{2}, \frac{-7+h}{2}\right)$

$M = \left(-3, \frac{h-7}{2}\right)$

Step 2:

gradient $MX = \frac{\frac{-13}{2} - \left(\frac{h-7}{2}\right)}{h-(-3)} = \frac{-13-(h-7)}{2(h+3)}$

$= \frac{-6-h}{2h+6}$

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Continuation of working space for question **13(b)**.

Step 3 :

$$\frac{-6-h}{2h+6} = \frac{-10}{h+7}$$

$$(-6-h)(h+7) = -10(2h+6)$$

$$-6h-42-h^2-7h = -20h-60$$

$$h^2-7h-18=0$$

$$(h+2)(h-9)=0$$

$$\therefore h=-2 \text{ or } 9$$

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