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MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CENTRE NUMBER S

INDEX NUMBER

MATHEMATICS 9758/01

Paper 1

October/November 2023

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **22** printed pages and **2** blank pages.

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1 Find the exact equation of the tangent to the curve

$$\ln y = (11 - 5x)^2$$

at the point where $x = 2$. [5]

Step 1: When $x = 2$, $\ln y = (11 - 10)^2$

$\therefore y = e$

Step 2: Differentiate w.r.t x $\ln y = (11 - 5x)^2$

$\frac{1}{y}\frac{dy}{dx} = 2(11 - 5x)(-5)$

$\frac{1}{y}\frac{dy}{dx} = -10(11 - 5x)$

when $x = 2, y = e$: $\frac{1}{e}\frac{dy}{dx} = -10(11 - 10)$

$\frac{dy}{dx} = -10e$

Step 3: Equation of tangent:

$y - e = -10e(x - 2)$

$y - e = -10ex + 20e$

$y = -10ex + 21e$

2 The first four terms of a sequence are given by $u_1 = 10$, $u_2 = 61$, $u_3 = 206$ and $u_4 = 469$. It is given that u_n is a cubic polynomial.

(a) Find u_n in terms of n. [3]

Let $u_n = an^3 + bn^2 + cn + d$

$10 = a + b + c + d$ — (1)

$61 = 8a + 4b + 2c + d$ — (2)

$206 = 27a + 9b + 3c + d$ — (3)

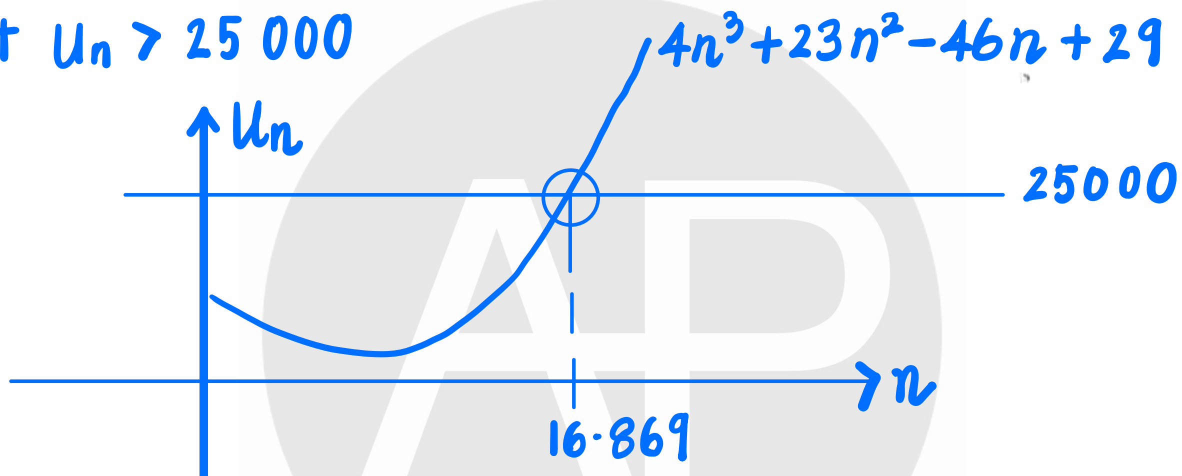
$469 = 64a + 16b + 4c + d$ — (4)

By GC: $a = 4$, $b = 23$, $c = -46$, $d = 29$

$\therefore u_n = 4n^3 + 23n^2 - 46n + 29$

(b) Find the range of values of n for which u_n is greater than 25 000. [2]

Let $u_n > 25\,000$



By GC: $n \geq 17$, $n \in \mathbb{Z}^+$

3 Vectors **a** and **b** are such that **a.b** $= -1$. It is also given that $(\mathbf{a}\times\mathbf{b}+\mathbf{a})$ is perpendicular to $(\mathbf{a}\times\mathbf{b}+\mathbf{b})$.

(a) Show that $|\mathbf{a}\times\mathbf{b}| = 1$. [3]

$$[(\underset{\sim}{a}\times\underset{\sim}{b})+\underset{\sim}{a}]\cdot[(\underset{\sim}{a}\times\underset{\sim}{b})+\underset{\sim}{b}] = 0$$

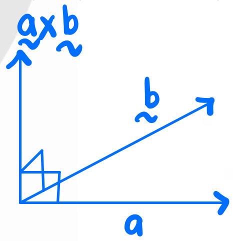
Recap:
If $\underset{\sim}{a} \perp \underset{\sim}{b}$,
then $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$

$$|\underset{\sim}{a}\times\underset{\sim}{b}|^2 + (\underset{\sim}{a}\times\underset{\sim}{b})\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot(\underset{\sim}{a}\times\underset{\sim}{b}) + \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$$

$$|\underset{\sim}{a}\times\underset{\sim}{b}|^2 + 0 + 0 - 1 = 0$$

$$\therefore |\underset{\sim}{a}\times\underset{\sim}{b}|^2 = 1$$

$$|\underset{\sim}{a}\times\underset{\sim}{b}| = 1 \text{ (shown)}$$



$(\underset{\sim}{a}\times\underset{\sim}{b})$ is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(b) Hence find the angle between the direction of **a** and the direction of **b**. [3]

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = -1 \qquad |\underset{\sim}{a}\times\underset{\sim}{b}| = 1$$

$$|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta = -1 \quad \text{--- (1)} \qquad |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta = 1 \quad \text{--- (2)}$$

$$\frac{(2)}{(1)}: \tan\theta = -1$$

$$\theta = 135^\circ \text{ (obtuse angle)}$$

Recap:
1. $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$
2. $\underset{\sim}{a}\times\underset{\sim}{b} = (|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta)\,\hat{\underset{\sim}{n}}$
3. $|\underset{\sim}{a}\times\underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$

4 (a) Find $\int \cos px \cos qx \, dx$, where p and q are constants such that $p \neq q$ and $p \neq -q$. [2]

$$\int \cos px \cos qx \, dx$$
$$= \frac{1}{2}\int 2\cos px \cos qx \, dx$$
$$= \frac{1}{2}\int \cos(p+q)x + \cos(p-q)x \, dx$$
$$= \frac{1}{2}\left[\frac{\sin(p+q)x}{(p+q)} + \frac{\sin(p-q)x}{(p-q)}\right] + C$$

Factor Formula:

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

Let $\frac{A+B}{2} = px$

$A + B = 2px$ —— ①

Let $\frac{A-B}{2} = qx$

$A - B = 2qx$ —— ②

Solving simultaneously,

$A = (p+q)x$, $B = (p-q)x$

(b) Given that $n \neq 0$, show that $\int x\cos nx \, dx = \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} + c$, where c is an arbitrary constant. [3]

$$\int x\cos nx \, dx = x \cdot \frac{\sin nx}{n} - \int \frac{\sin nx}{n} \, dx$$
$$= \frac{x\sin nx}{n} - \left[\frac{-\cos nx}{n^2}\right] + C$$
$$= \frac{x\sin nx}{n} + \frac{\cos nx}{n^2} + C$$

Integration by Parts:

Let $u = x$

$\frac{du}{dx} = 1$

Let $dv = \cos nx$

$v = \frac{\sin nx}{n}$

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(c) Using the result in part **(b)** show that, for all positive integers n, the value of $\int_0^{\pi} x\cos nx\,dx$ can be expressed as $\frac{k}{n^2}$, where the possible value(s) of k are to be determined. [2]

$$\int_0^{\pi} x\cos nx\,dx = \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

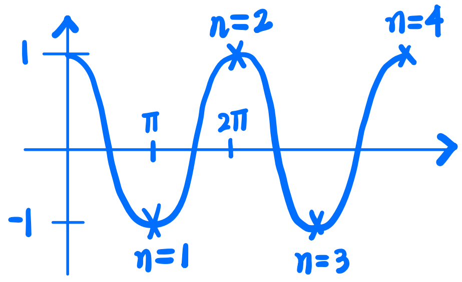
$$= \frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos 0}{n^2}$$

$$= 0 + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}$$

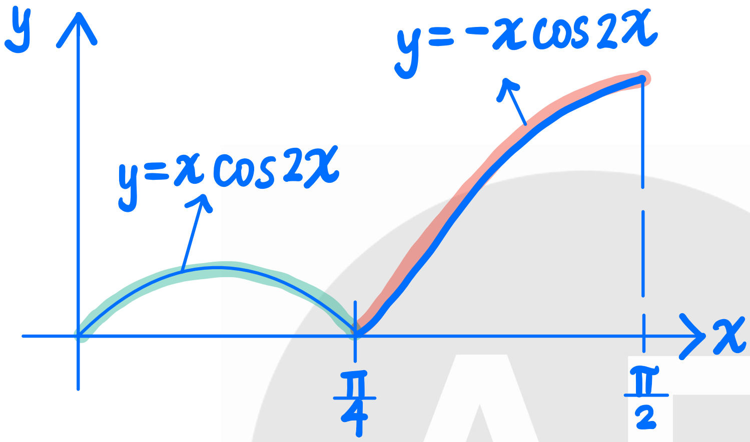
$$= \frac{\cos n\pi - 1}{n^2}$$

$$= \underline{\frac{-2}{n^2} \text{ or } \frac{0}{n^2}}$$

(when n is odd) (when n is even)



(d) Using the result in part **(b)** find the exact value of $\int_0^{\frac{\pi}{2}} |x\cos 2x|\,dx$. [3]



from part ⓑ :

s $nx\,dx$

$$= \frac{x\sin nx}{n} + \frac{\cos nx}{n^2} + C$$

$$\int_0^{\frac{\pi}{2}} |x\cos 2x|$$

$$= \int_0^{\frac{\pi}{4}} x\cos 2x\,dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\cos 2x\,dx$$

$$= \left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{2^2}\right]_0^{\frac{\pi}{4}} - \left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{2^2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{8} + 0 - 0 - \frac{1}{4}\right] - \left[0 - \frac{1}{4} - \frac{\pi}{8} - 0\right]$$

$$= \frac{\pi}{8} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} + \frac{\pi}{8}$$

$$= \underline{\frac{\pi}{4} \text{ units}^2}$$

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5 (a) Use the method of differences to show that $\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \ln\left(\frac{n+1}{n}\right) - \ln 2.$ [3]

$$\sum_{r=2}^{n} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \sum_{r=2}^{n} \ln(r-1) + \ln(r+1) - 2\ln r$$

$$\begin{aligned}
= &\ (\ln 1 + \ln 3 - 2\ln 2) \\
&+ (\ln 2 + \ln 4 - 2\ln 3) \\
&+ (\ln 3 + \ln 5 - 2\ln 4) \\
&+ (\ln 4 + \ln 6 - 2\ln 5) \\
&+ (\ln 5 + \ln 7 - 2\ln 6) \\
&+ \ \vdots \\
&+ [\ln(n-2) + \ln n - 2\ln(n-1)] \\
&+ [\ln(n-1) + \ln(n+1) - 2\ln n]
\end{aligned}$$

$$= \ln 1 - 2\ln 2 + \ln 2 + \ln n + \ln(n+1) - 2\ln n$$

$$= -\ln 2 - \ln n + \ln(n+1)$$

$$= \ln\left(\frac{n+1}{n}\right) - \ln 2$$

(b) Show that the corresponding infinite series is convergent and state the sum to infinity. [2]

$\ln\left(\frac{n+1}{n}\right) = \ln\left(1+\frac{1}{n}\right)$

As $n \to \infty$, $\ln\left(1+\frac{1}{n}\right) \to \ln 1$
$= 0$

Hence, as $n \to \infty$, $\ln\left(\frac{n+1}{n}\right) - \ln 2 \to -\ln 2$.

The infinite series is convergent as it converges to a constant, i.e. $-\ln 2$.

$\therefore \underline{S_\infty = -\ln 2}$

(c) Show that $\sum_{r=10}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \ln\left(\frac{a}{b}\right)$, where a and b are integers to be found. [2]

$$\sum_{r=10}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] = \sum_{r=2}^{20} \ln\left[\frac{(r-1)(r+1)}{r^2}\right] - \sum_{r=2}^{9} \ln\left[\frac{(r-1)(r+1)}{r^2}\right]$$

$$= \left(\ln \frac{21}{20} - \cancel{\ln 2}\right) - \left(\ln \frac{10}{9} - \cancel{\ln 2}\right)$$

$$= \ln\left[\frac{21}{20} \div \frac{10}{9}\right]$$

$$= \ln\left[\frac{21}{20} \times \frac{9}{10}\right]$$

$$= \ln\left(\frac{189}{200}\right)$$

where $\underline{a = 189,\ b = 200}$

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6 **(a)** Using double angle formulae, show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [2]

$$\cos^4\theta = (\cos^2\theta)^2$$
$$= \left(\frac{\cos 2\theta + 1}{2}\right)^2$$
$$= \frac{1 + 2\cos 2\theta + \cos^2 2\theta}{4}$$
$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \left[\frac{\cos 4\theta + 1}{2}\right]\right)$$
$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right)$$
$$= \frac{1}{8}(2 + 4\cos 2\theta + 1 + \cos 4\theta)$$
$$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

Double Angle Formula

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore \cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

(b) The region R lies in the first quadrant and is bounded by the curve $y^4 = (9 - x^2)^3$, the x-axis and the lines $x = 1.5$ and $x = 3$. R is rotated about the x-axis through 2π radians. Using the substitution $x = 3\sin\theta$, find the exact volume generated. [6]

$$R = \pi\int_{x=1.5}^{x=3} y^2\,dx$$
$$= \pi\int_{1.5}^{3} (9 - x^2)^{\frac{3}{2}}\,dx$$
$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (9 - 9\sin^2\theta)^{\frac{3}{2}} \cdot 3\cos\theta\,d\theta$$
$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9^{\frac{3}{2}}(\cos^2\theta)^{\frac{3}{2}} \cdot 3\cos\theta\,d\theta$$
$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 81\cos^4\theta\,d\theta$$
$$= \frac{81\pi}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos 4\theta + 4\cos 2\theta + 3)\,d\theta$$
$$= \frac{81\pi}{8}\left[\frac{\sin 4\theta}{4} + \frac{4\sin 2\theta}{2} + 3\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= \frac{81\pi}{8}\left[\left(0 + 0 + \frac{3\pi}{2}\right) - \left(\frac{\sqrt{3}}{8} + \sqrt{3} + \frac{\pi}{2}\right)\right]$$
$$= \frac{81\pi}{8}\left(\pi - \frac{9\sqrt{3}}{8}\right) \text{ units}^2$$

$x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

when $x = 1.5$

$1.5 = 3\sin\theta$

$\sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$

when $x = 3$

$3 = 3\sin\theta$

$\sin\theta = 1$

$\therefore \theta = \frac{\pi}{2}$

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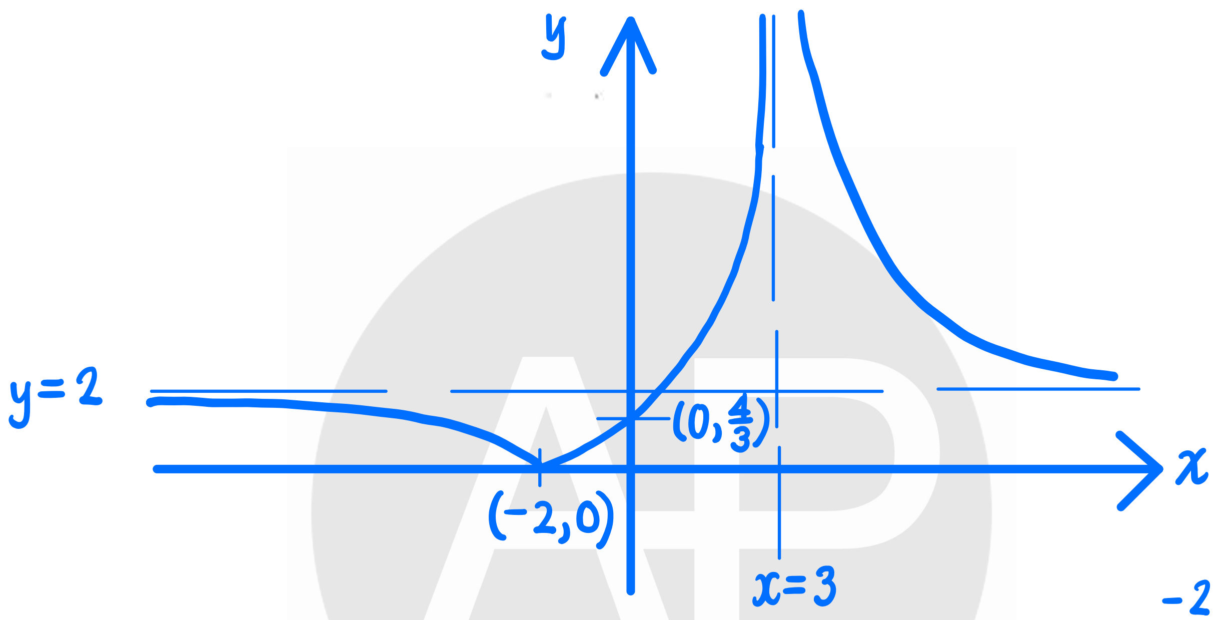
6 [Continued]

7 The function f is defined by

$$\mathrm{f}: x \mapsto \left|\frac{2x+4}{3-x}\right|, \qquad x \in \mathbb{R}, \qquad x \neq 3.$$

(a) Sketch the graph of $y = \mathrm{f}(x)$, giving the equations of any asymptotes and the coordinates of the points where the curve meets the axes. [3]



When $y=0$, $2x+4=0$

$\underline{x=-2}$

When $x=0$, $\underline{y=\frac{4}{3}}$

$$\begin{array}{r} -2 \\ -x+3\,\overline{)\,2x+4} \\ \underline{-(2x-6)} \\ 10 \end{array}$$

$\mathrm{f}: x \to \left|-2 + \frac{10}{3-x}\right|$

as $x \to \pm\infty$,

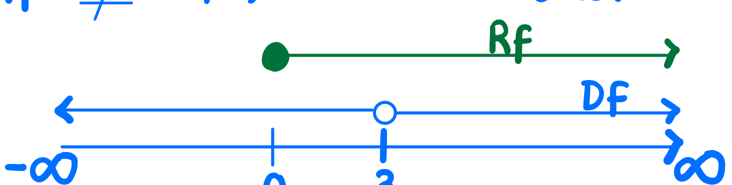
$\mathrm{f}: x \to |-2| = 2$

(b) Hence state the range of f. [1]

$R_f = [0, \infty)$

(c) Explain why the function f^2 does not exist. [1]

Since $R_f \not\subseteq D_f$, f^2 does not exist.



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The domain of f is further restricted to $x \leqslant a$, where a is a constant.

(d) State the greatest value of a such that the function f^{-1} exists. [1]

$a = -2$

(e) Hence find $\mathrm{f}^{-1}(x)$ and state its domain. [4]

When $x \leqslant -2$,

$$f(x) = -\left(\frac{2x+4}{3-x}\right)$$

$$\therefore \quad y = \frac{-2x-4}{3-x}$$

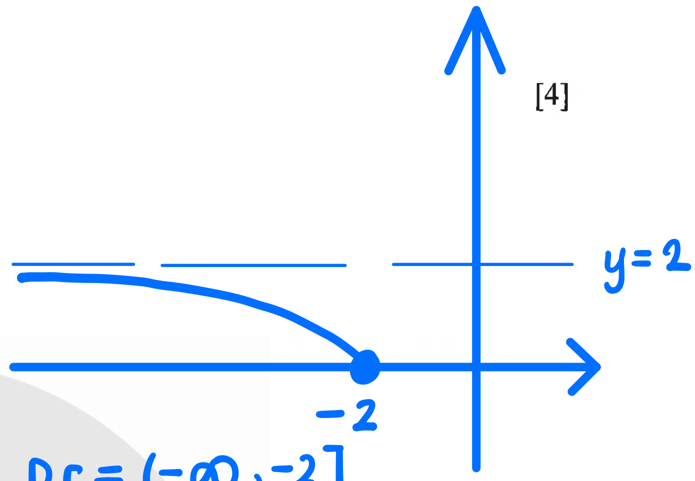
$$3y - xy = -2x - 4$$

$$2x - xy = -4 - 3y$$

$$x = \frac{-4-3y}{2-y}$$

$$= \frac{4+3y}{y-2}$$

$$\therefore f^{-1}(x) = \frac{4+3x}{x-2}, \quad 0 \leqslant x < 2$$



$D_f = (-\infty, -2]$

$R_f = [0, 2)$

$$\boxed{D_{f^{-1}} = R_f}$$

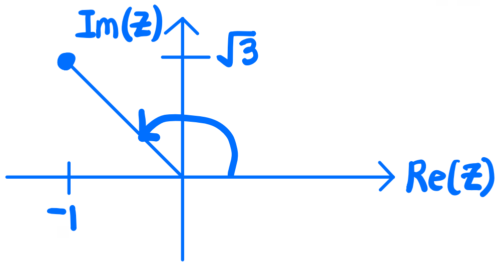
8 Do not use a calculator in answering this question.

(a) (i) Express z, where $z = -1 + \sqrt{3}\mathrm{i}$, in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [2]

$\arg z = \pi - \tan^{-1}\sqrt{3}$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$



$|z| = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\therefore z = 2e^{i\frac{2\pi}{3}}$

(ii) Find the smallest positive integer value of n such that $\frac{z^n}{\mathrm{i}z^*}$ is purely imaginary. [3]

$\left|\frac{z^n}{iz^*}\right| = \frac{|z|^n}{|z^*|} = \frac{2^n}{2} = 2^{n-1}$

$\arg\left(\frac{z^n}{iz^*}\right) = \arg z^n - \arg iz^*$

$= n\arg z - [\arg i + \arg z^*]$

$= \frac{2n\pi}{3} - \frac{\pi}{2} - \left(-\frac{2\pi}{3}\right)$

$= \frac{4n\pi - 3\pi + 4\pi}{6}$

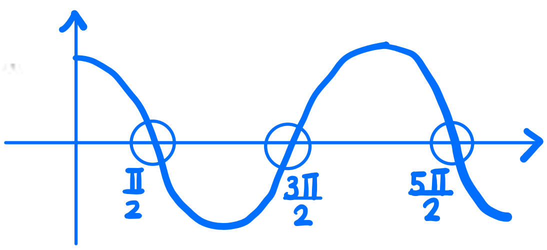
$= \frac{\pi}{6}(4n+1)$

$\therefore \frac{z^n}{iz^*} = 2^{n-1}\left[\cos\frac{\pi}{6}(4n+1) + i\sin\frac{\pi}{6}(4n+1)\right]$

For $\frac{z^n}{iz^*}$ to be purely imaginary,

$\cos\frac{\pi}{6}(4n+1) = 0$.

By trial and error, smallest positive $n = 2$



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(b) Find the complex numbers v and w which satisfy the following simultaneous equations.

$$2v+|w|=1$$
$$3v-\mathrm{i}w = -3+4\mathrm{i}$$

Give your answers in the form $a+\mathrm{i}b$, where a and b are real numbers. [5]

$2v+|w|=1$ —— (1)

$3v-\mathrm{i}w = -3+4\mathrm{i}$ —— (2)

from (1): $v = \dfrac{1-|w|}{2}$ —— (3)

Sub (3) into (2):

$3\left[\dfrac{1-|w|}{2}\right] - iw = -3+4i$

$3-3|w|-2iw = -6+8i$

Let $w = a+bi$:

$3-3|a+bi|-2i(a+bi) = -6+8i$

$3-3\sqrt{a^2+b^2}-2ai+2b = -6+8i$

$(3-3\sqrt{a^2+b^2}+2b+6)+(-2a-8)i = 0+0i$

by comparison of coefficients:

$-2a-8=0$

$2a=-8$

$a=-4$

$\therefore\ 3-3\sqrt{16+b^2}+2b+6=0$

$3\sqrt{16+b^2} = 2b+9$

$9(16+b^2) = 4b^2+36b+81$

$5b^2-36b+63=0$

$(5b-21)(b-3)=0$

$b=\dfrac{21}{5}$ or 3

When $w=-4+\frac{21}{5}i$, $v = \dfrac{1-\left|-4+\frac{21}{5}i\right|}{2} = \dfrac{-12}{5}$

When $w=-4+3i$, $v = \dfrac{1-|-4+3i|}{2} = -2$

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9 The line l_1 contains the point A with coordinates $(-3, 1, 2)$ and is parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$, where a is a constant. The line l_2 has equation $\frac{x+2}{3} = \frac{y-1}{2} = z-5$. It is given that l_1 and l_2 cross at the point B.

(a) Find the value of a and the coordinates of B. [5]

Step 1:

$l_1: \underset{\sim}{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}, \lambda \in \mathbb{R}$

$l_2: \underset{\sim}{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$

Step 2:

Let $\begin{pmatrix} -3+2\lambda \\ 1+\lambda \\ 2+a\lambda \end{pmatrix} = \begin{pmatrix} -2+3\mu \\ 1+2\mu \\ 5+\mu \end{pmatrix}$

$-3+2\lambda = -2+3\mu$

$2\lambda = 1+3\mu$ — (1)

$1+\lambda = 1+2\mu$

$\lambda = 2\mu$ — (2)

$2+a\lambda = 5+\mu$ — (3)

Sub (2) into (1): $2(2\mu) = 1+3\mu$

$\mu = 1$

$\therefore \lambda = 2$

Sub $\mu = 1, \lambda = 2$ into (3):

$2+2a = 5+1$

$2a = 4$

$a = 2$

$\therefore \overrightarrow{OB} = \begin{pmatrix} -3+2(2) \\ 1+2 \\ 2+2(2) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$

coordinates of $B = (1, 3, 6)$

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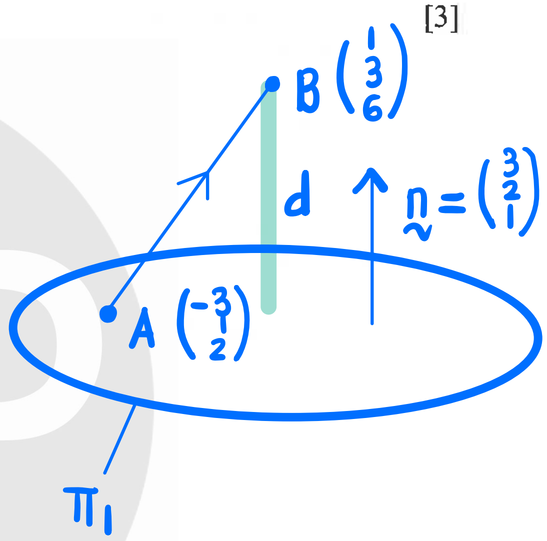
(b) The plane π_1 contains the point A and is perpendicular to l_2.

(i) Find the shortest distance from B to π_1. [3]



Let the shortest distance of B to π_1 be d.

$$d = \left| \overrightarrow{AB} \cdot \widehat{\begin{pmatrix}3\\2\\1\end{pmatrix}} \right|$$

$$= \left| \left[\begin{pmatrix}1\\3\\6\end{pmatrix} - \begin{pmatrix}-3\\1\\2\end{pmatrix}\right] \cdot \frac{\begin{pmatrix}3\\2\\1\end{pmatrix}}{\sqrt{14}} \right|$$

$$= \left| \frac{\begin{pmatrix}4\\2\\4\end{pmatrix} \cdot \begin{pmatrix}3\\2\\1\end{pmatrix}}{\sqrt{14}} \right|$$

$$= \frac{20}{\sqrt{14}} \text{ units} \quad (5.35 \text{ units})$$

(ii) Hence find the acute angle between l_1 and π_1. [2]

$$\theta = \sin^{-1} \left| \frac{\begin{pmatrix}2\\1\\2\end{pmatrix} \cdot \begin{pmatrix}3\\2\\1\end{pmatrix}}{\sqrt{9}\sqrt{14}} \right|$$

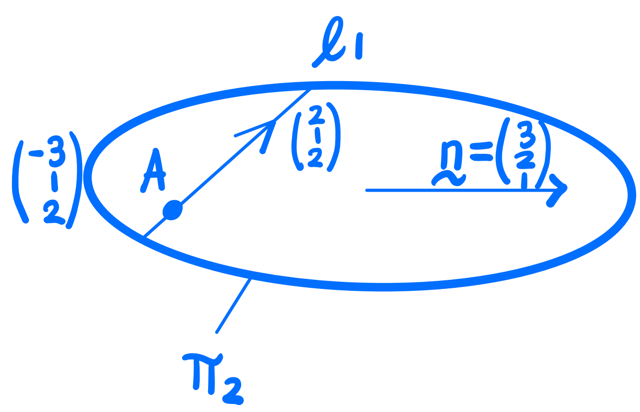
$$= \sin^{-1} \frac{10}{3\sqrt{14}}$$

$$= 63.0^\circ$$

(c) The plane π_2 is perpendicular to π_1 and contains l_1. Find a cartesian equation of π_2. [3]

$$\underset{\sim}{n_2} = \begin{pmatrix}2\\1\\2\end{pmatrix} \times \begin{pmatrix}3\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}-3\\4\\1\end{pmatrix}$$



$$\underset{\sim}{r} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix}$$

$$= 9 + 4 + 2$$

$$= 15$$

$$\therefore \pi_2 : -3x + 4y + z = 15$$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

10 The mass of a person depends both on daily rate of energy intake and on daily rate of energy expenditure.

In this question, mass is in kg, time is in days, and energy intake and energy expenditure are measured in Calories per day.

The rate of change of a person's mass with respect to time is proportional to the difference between energy intake and energy expenditure.

Andrew has a mass of M kg and his energy intake is fixed at C Calories per day. For every kg of his mass, he expends 30 Calories per day.

(a) Show that $\frac{dM}{dt} = k(C - 30M)$, where t is time and k is a constant. [1]

Since Andrew's mass is M kg, his energy expenditure per day $= 30 \times M = 30M$

Given $\frac{dM}{dt} \propto (C - 30M)$

$\therefore \frac{dM}{dt} = k(C - 30M)$, k is a constant.

Andrew's initial mass is 110 kg.

(b) Find the energy intake such that he maintains his mass at 110 kg. [1]

When $M = 110$ kg, $\frac{dM}{dt} = 0$

$\therefore k(C - 30(110)) = 0$

$C = \underline{3300 \text{ calories/day}}$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

As part of a new health plan, Andrew fixes his energy intake at 80% of the value found in part **(b)**.

(c) By solving the differential equation in part **(a)**, show that Andrew's mass while he is on the plan satisfies the equation $M = 88 + 22e^{-30kt}$. [4]

$C = 80\%$ of $3300 = \underline{2640}$ calories/day.

i.e. $\dfrac{dM}{dt} = k(2640 - 30M)$

$= 30k(88 - M)$

$\displaystyle\int \frac{1}{88-M}\, dM = 30k \int dt$

$-\ln|88-M| = 30kt + C$

$\ln|88-M| = -30kt - C$

$88 - M = \pm e^{-30kt-C}$

$= \pm e^{-C} \cdot e^{-30kt}$

$= Ae^{-30kt}$ (where $A = \pm e^{-C}$)

$\therefore M = 88 - Ae^{-30kt}$

<u>When $t = 0$, $M = 110$ kg</u>:

$110 = 88 - Ae^0$

$A = -22$

Hence, $\underline{M = 88 + 22e^{-30kt}}$

10 [Continued]

Andrew's mass after 75 days on the plan is 100 kg.

(d) Find the number of additional days required on the plan for Andrew's mass to fall below 96 kg. [4]

Step 1 :
When $t = 75$, $M = 100$ kg :

$100 = 88 + 22e^{-30(75k)}$

$12 = 22e^{-2250k}$

$e^{-2250k} = \frac{6}{11}$

$-2250k = \ln \frac{6}{11}$

$k = 0.00026939$

Step 2 :

$M = 88 + 22e^{-30(0.00026939)t}$

$96 = 88 + 22e^{-0.0080818t}$

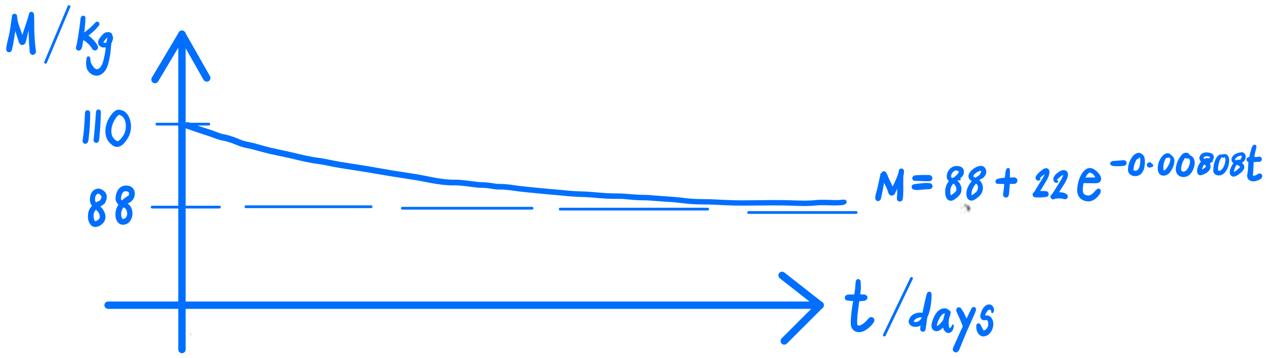
$e^{-0.0080818t} = \frac{4}{11}$

$-0.0080818t = \ln \frac{4}{11}$

$t = 125.17$ days

Hence, additional days $= 126 - 75$
$= \underline{51 \text{ days}}$

(e) **(i)** Sketch a graph of Andrew's mass while on this plan. Explain why Andrew cannot achieve a mass of 80 kg using this plan. [2]



As $t \to \infty$, $e^{-0.0080818t} \to 0$,
hence, $M \longrightarrow 88$ kg.
Andrew's mass will never go below 88 kg.

(ii) State the range of possible values of energy intake for which Andrew could achieve a mass of 80 kg. [1]

from part (c): $\frac{dM}{dt} = k(2640 - 30M)$
$= 30k(88 - M)$

by observation, $C_{max} = 30 \times 80$
$= 2400$

$\therefore \underline{0 < C < 2400}$

11 Wei is saving for a property that she intends to buy in the future. Wei needs to save a minimum of \$50 000. She saves regularly in an account which offers no interest. She makes an initial deposit on 31 January 2021 of \$$a$. Each subsequent month, she deposits \$50 more than she deposited in the previous month. Her final deposit is made on 31 December 2023.

(a) Find, to the nearest cent, the smallest value of a so that she saves at least \$50 000. [2]

a, $a+50$, $a+100$, …

T_1 T_2 T_3 T_{36}

31/01/21 31/12/23

$S_{36} \geq 50000$

$\frac{36}{2}(2a + 35(50)) \geq 50\,000$

$a \geq 513.89$ $\quad \therefore a_{min.} = \underline{\$513.89}$

Wei has arranged to purchase a property on 1 January 2024 with the aid of a loan of \$400 000. The terms of the loan are that interest of 0.1% is added to the amount owing at the start of every month, with the first interest amount added on 1 January 2024. Wei makes a monthly repayment of \$$x$ at the end of every month, with the first repayment on 31 January 2024.

(b) Show that the amount, in dollars, that Wei owes at the end of n months is

$$400000 \times 1.001^n - 1000x(1.001^n - 1).$$ [3]

n	amount, in dollars, that Wei owes at the end of n months
1	$(400\,000 \times 1.001) - x$
2	$[(400\,000 \times 1.001) - x] \times 1.001 - x$ $= 400000(1.001)^2 - 1.001x - x$
3	$[(400000(1.001)^2 - 1.001x - x] \times 1.001 - x$ $= 400000(1.001)^3 - 1.001^2x - 1.001x - x$
n	⋮ $400000(1.001)^n - 1.001^{n-1}x - 1.001^{n-2}x - \dots\dots - 1.001x - x$ $= 400000(1.001)^n - x\underbrace{[1 + 1.001 + 1.001^2 + \dots + 1.001^{n-1}]}_{\text{sum of GP with } n \text{ terms}}$ $= 400000(1.001)^n - x\left[\frac{1.001^n - 1}{1.001 - 1}\right]$ $= 400\,000(1.001^n) - 1000x(1.001^n - 1)$ (shown)

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

11 **[Continued]**

(c) **(i)** Suppose that the loan is repaid in 360 monthly repayments. Find the value of the monthly repayment to the nearest cent. [2]

Let $400\,000(1.001^{360}) - 1000x(1.001^{360} - 1) = 0$

$$x = \frac{400\,000(1.001^{360})}{1000(1.001^{360} - 1)}$$

$$= \underline{\$1323.63}$$

(ii) Use your answer to part **(c)(i)** to calculate the total interest paid on the loan in this case. [1]

$360(1323.634777) - 400\,000 = \$76\,508.51956$

$= \underline{\$76\,508.52}$

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

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(d) **(i)** Now suppose Wei decides to repay the loan at the rate of \$1600 per month for k months, plus a final monthly repayment of \$$y$, where $y < 1600$. Find k and y. [4]

Let $400\,000(1.001^k) - 1000(1600)(1.001^k - 1) < 0$

$400\,000(1.001^k) - 1\,600\,000(1.001^k - 1) < 0$

$1\,600\,000(1.001^k - 1) > 400\,000(1.001^k)$

$1.001^k - 1 > 0.25(1.001^k)$

$0.75(1.001^k) > 1$

$1.001^k > \frac{4}{3}$

$k > \frac{\ln \frac{4}{3}}{\ln 1.001}$

$k > 287.83$

$\therefore \underline{k = 287}$

Amt. Wei owes at end of 287 months

$= 400\,000(1.001^{287}) - 1000(1600)(1.001^{287} - 1)$

$= \$1320.22$

$\therefore \underline{y = \$1320.22}$

(ii) Hence find the total saving to Wei by making the repayments in part **(d)(i)** rather than the repayments in part **(c)(i)**, giving your answer to 4 significant figures. [1]

$360(1323.634777) - [(1600 \times 287) + 1320.217991]$

$= \$15\,988.30173$

$= \underline{\$15\,990}$ (4 s.f.)

The suggested solutions are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.