1 A function f is defined by $f(x) = ax^3 + bx^2 + cx + d$. The graph of $y = f(x)$ passes through the points $(1, 5)$ and $(-1, -3)$. The graph has a turning point at $x = 1$, and $\int_0^1 f(x)\,dx = 6$.

Find the values of a, b, c and d. [5]

Sub (1,5) into f(x): $5 = a + b + c + d$ — 1

Sub (-1,-3) into f(x): $-3 = -a + b - c + d$ — 2

$f'(x) = 3ax^2 + 2bx + c$

when $x = 1$, $f'(x) = 0$: $0 = 3a + 2b + c$ — 3

Given $\int_0^1 f(x)\,dx = 6$

$$\left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_0^1 = 6$$

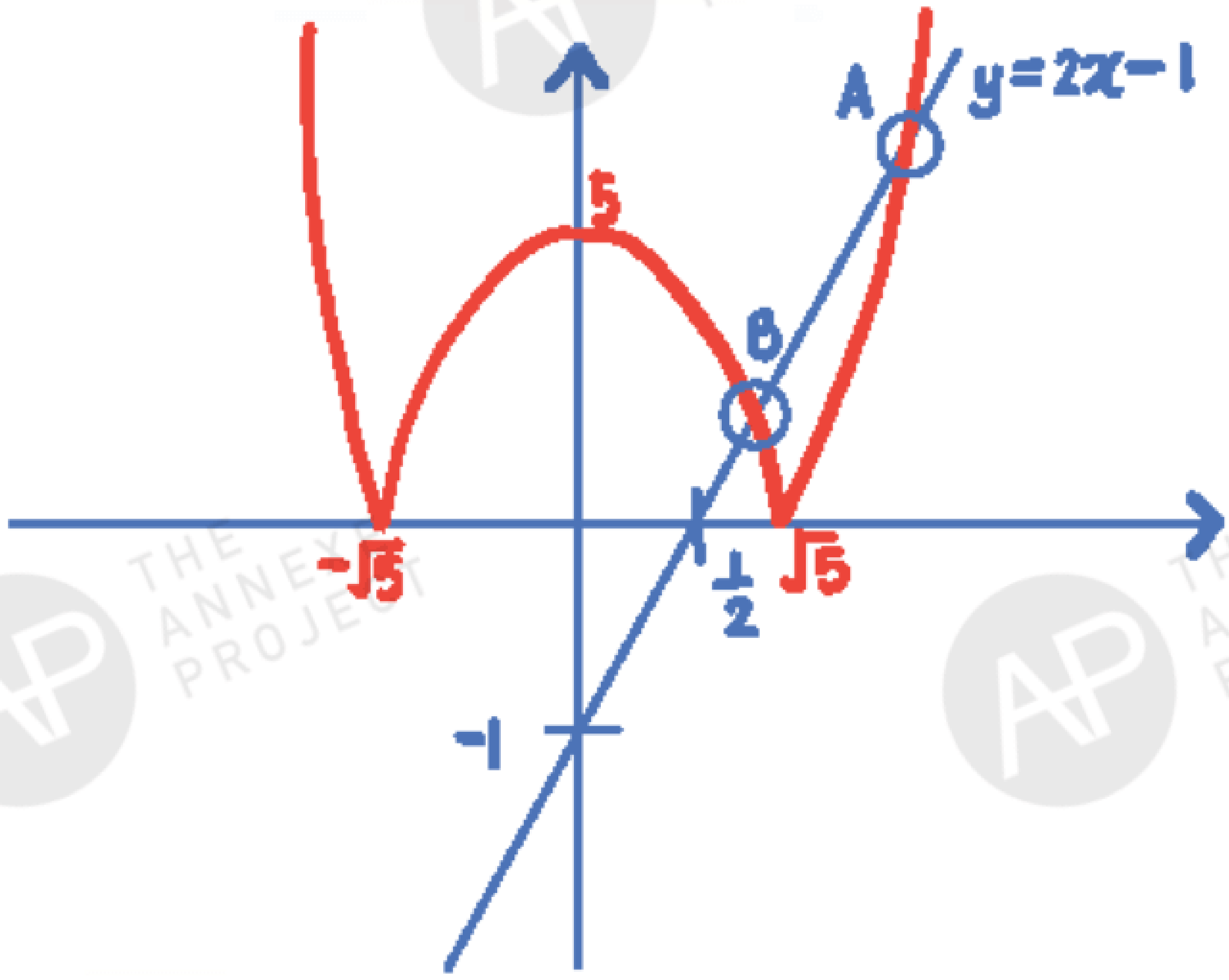
$$\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 6$$

$$3a + 4b + 6c + 12d = 72 \quad \text{— 4}$$

By GC: $a = 4$, $b = -6$, $c = 0$, $d = 7$

DO NOT WRITE IN THIS MARGIN

2 (a) Sketch, on the same axes, the graphs of $y = |x^2 - 5|$ and $y = 2x - 1$. [2]



(b) Find the exact solutions of $|x^2 - 5| = 2x - 1$. [4]

Finding x-coordinate of A:

Let $x^2 - 5 = 2x - 1$

$x^2 - 2x - 4 = 0$

$x = \dfrac{2 \pm \sqrt{4 - 4(-4)}}{2}$

$= \dfrac{2 \pm 2\sqrt{5}}{2}$

$= 1 \pm \sqrt{5}$

$= \underline{1 + \sqrt{5}}$

$(\because x > \sqrt{5})$

Finding x-coordinate of B:

Let $5 - x^2 = 2x - 1$

$x^2 + 2x - 6 = 0$

$x = \dfrac{-2 \pm \sqrt{4 - 4(-6)}}{2}$

$= \dfrac{-2 \pm 2\sqrt{7}}{2}$

$= -1 \pm \sqrt{7}$

$= \underline{-1 + \sqrt{7}}$

$(\because x > \frac{1}{2})$

3 A curve has equation $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3$, for $x > 0$ and $y > 0$.

(a) Show that $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{2}}$. [2]

Given $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3$

differentiate both sides w.r.t x:

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$$= -\left(\frac{y}{x}\right)^{\frac{1}{2}}$$

(b) Find the equation of the normal to the curve at the point where $x = 1$. [4]

When $x=1$, $1+\sqrt{y} = 3$, $\frac{dy}{dx} = -\left(\frac{4}{1}\right)^{\frac{1}{2}}$

$\sqrt{y} = 2$ $= -2$

$y = 4$

$\therefore$ gradient of normal $= \frac{1}{2}$

equation of normal: $y-4 = \frac{1}{2}(x-1)$

$y = \frac{1}{2}x + \frac{7}{2}$

4 **Do not use a calculator in answering this question.**

The complex number z is given by

$$z = \frac{\left(\cos\left(\frac{1}{16}\pi\right) + \mathrm{i}\sin\left(\frac{1}{16}\pi\right)\right)^2}{\cos\left(\frac{1}{8}\pi\right) - \mathrm{i}\sin\left(\frac{1}{8}\pi\right)}.$$

(a) Find $|z|$ and $\arg(z)$. Hence find the value of z^2. [3]

Given $z = \dfrac{(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16})^2}{\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}} = \dfrac{(e^{i\frac{\pi}{16}})^2}{(e^{-i\frac{\pi}{8}})} = \dfrac{e^{i\frac{\pi}{8}}}{e^{-i\frac{\pi}{8}}} = e^{i\frac{\pi}{8}+i\frac{\pi}{8}} = e^{i\frac{\pi}{4}}$

$\underline{|z| = 1, \ \arg z = \frac{\pi}{4}}$

$z^2 = (e^{i\frac{\pi}{4}})^2 = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \underline{i}$

(b) (i) Show that

$$(\cos\theta + i\sin\theta)(1+\cos\theta - i\sin\theta) = 1 + \cos\theta + i\sin\theta.$$

[2]

LHS $= \cos\theta + \cos^2\theta - \cancel{i\sin\theta\cos\theta} + i\sin\theta + \cancel{i\sin\theta\cos\theta} + \sin^2\theta$

$= 1 + \cos\theta + i\sin\theta$

$=$ RHS

(ii) Hence, or otherwise, find the value of $(1+z)^4 + (1+z^*)^4$. [2]

from b(i): $z(1+z^*) = 1+z$

hence, $(1+z)^4 + (1+z^*)^4$

$= [z(1+z^*)]^4 + (1+z^*)^4$

$= z^4(1+z^*)^4 + (1+z^*)^4$

$= (1+z^*)^4(1+z^4)$

$= (1+z^*)^4(0)$

$= \underline{0}$

from part(a): Since $z^2 = i$, then $z^4 = i^2 = -1$

5 (a) Express $\frac{x}{(x+2)(x+3)(x+4)}$ in partial fractions. [3]

let $\frac{x}{(x+2)(x+3)(x+4)} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{x+4}$

$= \frac{A(x+3)(x+4) + B(x+2)(x+4) + C(x+2)(x+3)}{(x+2)(x+3)(x+4)}$

comparing the numerators:

Sub $x=-3$: $-3 = B(-1)(1)$

$\therefore B = 3$

Sub $x=-2$: $-2 = A(1)(2)$

$\therefore A = -1$

Sub $x=-4$: $-4 = C(-2)(-1)$

$\therefore C = -2$

hence, $\frac{x}{(x+2)(x+3)(x+4)} = \frac{-1}{(x+2)} + \frac{3}{(x+3)} - \frac{2}{(x+4)}$

(b) Hence find, in terms of n, $\sum_{r=1}^{n} \frac{r}{(r+2)(r+3)(r+4)}$. [3]

$$\sum_{r=1}^{n} \frac{-1}{r+2} + \frac{3}{r+3} - \frac{2}{r+4}$$

$$= \left(\frac{-1}{3} + \frac{3}{4} - \cancel{\frac{2}{5}}\right)$$
$$+ \left(\frac{-1}{4} + \cancel{\frac{3}{5}} - \cancel{\frac{2}{6}}\right)$$
$$+ \left(\cancel{\frac{-1}{5}} + \cancel{\frac{3}{6}} - \cancel{\frac{2}{7}}\right)$$
$$+ \left(\cancel{\frac{-1}{6}} + \cancel{\frac{3}{7}} - \cancel{\frac{2}{8}}\right)$$
$$+ \vdots$$
$$+ \left(\cancel{\frac{-1}{n+1}} + \cancel{\frac{3}{n+2}} - \frac{2}{n+3}\right)$$
$$+ \left(\cancel{\frac{-1}{n+2}} + \frac{3}{n+3} - \frac{2}{n+4}\right)$$

$$= \frac{-1}{3} + \frac{3}{4} - \frac{1}{4} - \frac{2}{n+3} + \frac{3}{n+3} - \frac{2}{n+4}$$

$$= \underline{\frac{1}{6} + \frac{1}{n+3} - \frac{2}{n+4}}$$

(c) State the value of $\sum_{r=1}^{\infty} \frac{r}{(r+2)(r+3)(r+4)}$. [1]

As $n \to \infty$, $\frac{1}{n+3} \to 0$, $\frac{2}{n+4} \to 0$

$$\therefore \frac{1}{6} + \frac{1}{n+3} - \frac{2}{n+4} \to \underline{\frac{1}{6}}$$

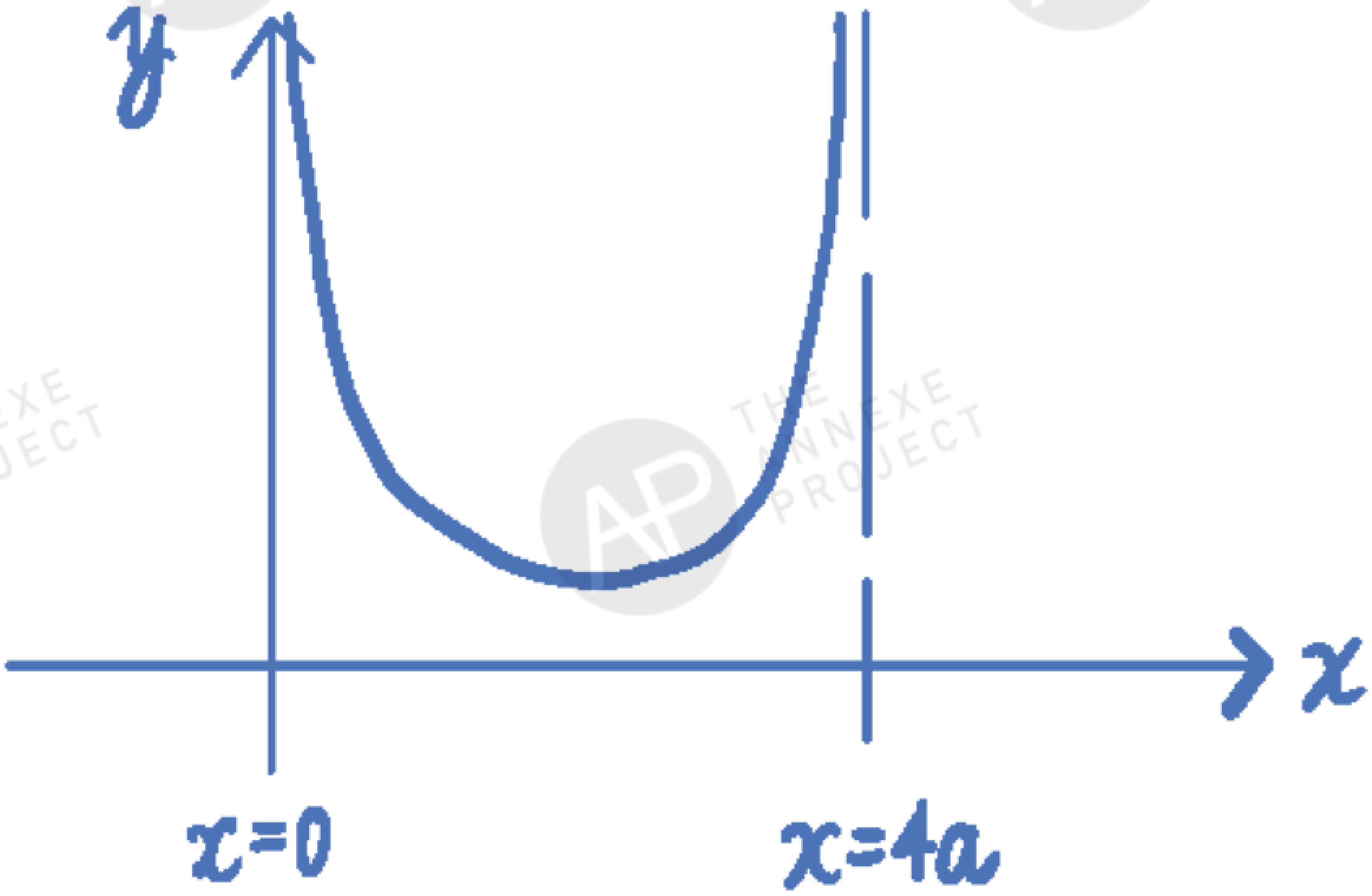
6 A curve C has equation $y = \dfrac{1}{\sqrt{4ax - x^2}}$, where $a > 0$.

(a) Sketch C and give the equations of any asymptotes, in terms of a where appropriate. [4]

$$y = \frac{1}{\sqrt{x(4a - x)}}$$

Vertical asymptotes:

$x = 0$ and $x = 4a$



(b) Find the smallest possible value of y in terms of a. [1]

When $x = 2a$,

$$y = \frac{1}{\sqrt{2a(4a-2a)}} = \frac{1}{2a}$$

(c) Describe the transformation that maps the graph of C onto the graph of $y = \frac{1}{\sqrt{4a^2 - x^2}}$. [3]

$$y = \frac{1}{\sqrt{4ax - x^2}} \xrightarrow{x \to x+2a} y = \frac{1}{\sqrt{4a(x+2a)-(x+2a)^2}}$$

$$= \frac{1}{\sqrt{\cancel{4ax} + 8a^2 - x^2 - \cancel{4ax} - 4a^2}}$$

$$= \frac{1}{\sqrt{4a^2 - x^2}}$$

translation of $2a$ units in the negative x-direction.

7 It is given that $y = e^{\sin^{-1} x}$, for $-1 < x < 1$.

(a) Show that $(1 - x^2)\dfrac{d^2 y}{dx^2} = x\dfrac{dy}{dx} + y$. [4]

Given $y = e^{\sin^{-1} x}$

$\therefore \ln y = \sin^{-1} x$

differentiate both sides w.r.t. x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,\frac{dy}{dx} = y \quad \text{———} \quad (1)$$

differentiate both sides w.r.t. x:

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \frac{dy}{dx}$$

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x\frac{dy}{dx}}{\sqrt{1-x^2}} = \frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = \sqrt{1-x^2}\,\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = y \quad \text{(from part (1))}$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y \quad \text{(shown).}$$

(b) Find the first 4 terms of the Maclaurin expansion of $e^{\sin^{-1}x}$. [5]

differentiate w.r.t. x for part ⓐ:

$$(1-x^2)\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}(-2x) = x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx}$$

When $x=0$,

$f(0) = e^{\sin^{-1}0} = 1$

$f'(0) = \frac{1}{\sqrt{1-0}} = 1$

$f^2(0) = 0+1 = 1$

$f^3(0) = 1+1 = 2$

$\therefore e^{\sin^{-1}x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$

8 The lines l_1 and l_2 have equations

$$\mathbf{r}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

respectively, where λ and μ are parameters.

(a) Find a cartesian equation of the plane containing l_1 and the point $(1, -3, -1)$. [4]

another direction vector $= \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$

$\underset{\sim}{n} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -16 \end{pmatrix} = 8\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$= 1 + 2$

$= 3$

Cartesian eqn of plane: $\underline{x - 2z = 3}$

(b) Show that l_1 is perpendicular to l_2. [2]

$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 2 - 6 + 4 = 0$

$\therefore$ l_1 is perpendicular to l_2.

15 $\mathbf{r}_1 = \begin{pmatrix}3\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\1\end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix}-4\\1\\-3\end{pmatrix} + \mu\begin{pmatrix}1\\2\\4\end{pmatrix}$

(c) (i) Find values of λ and μ such that $\mathbf{r}_1 - \mathbf{r}_2$ is perpendicular to both l_1 and l_2. State the position vectors of the points where the common perpendicular meets l_1 and l_2. [6]

$$\underset{\sim}{r}_1 - \underset{\sim}{r}_2 = \begin{pmatrix}3+2\lambda\\2-3\lambda\\\lambda\end{pmatrix} - \begin{pmatrix}-4+\mu\\1+2\mu\\-3+4\mu\end{pmatrix}$$

$$= \begin{pmatrix}7+2\lambda-\mu\\1-3\lambda-2\mu\\3+\lambda-4\mu\end{pmatrix}$$

$$\begin{pmatrix}7+2\lambda-\mu\\1-3\lambda-2\mu\\3+\lambda-4\mu\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix} = 0$$

$$14+4\lambda-2\mu-3+9\lambda+6\mu+3+\lambda-4\mu=0$$

$$14\lambda+14=0$$

$$\underline{\lambda=-1}$$

$$\begin{pmatrix}7+2\lambda-\mu\\1-3\lambda-2\mu\\3+\lambda-4\mu\end{pmatrix}\cdot\begin{pmatrix}1\\2\\4\end{pmatrix} = 0$$

$$7+2\lambda-\mu+2-6\lambda-4\mu+12+4\lambda-16\mu=0$$

$$-21\mu+21=0$$

$$\underline{\mu=1}$$

When $\lambda=-1$:
intersection betw. r_1-r_2 and $l_1 = \begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}2\\-3\\1\end{pmatrix} = \underline{\begin{pmatrix}1\\5\\-1\end{pmatrix}}$

When $\mu=1$:
intersection betw. r_1-r_2 and $l_2 = \begin{pmatrix}-4\\1\\-3\end{pmatrix} + \begin{pmatrix}1\\2\\4\end{pmatrix} = \underline{\begin{pmatrix}-3\\3\\1\end{pmatrix}}$

(ii) Find the length of this common perpendicular. [2]

$$\begin{pmatrix}1\\5\\-1\end{pmatrix} - \begin{pmatrix}-3\\3\\1\end{pmatrix} = \begin{pmatrix}4\\2\\-2\end{pmatrix}$$

$$\left|\begin{pmatrix}4\\2\\-2\end{pmatrix}\right| = \sqrt{4^2+2^2+(-2)^2}$$

$$= \sqrt{24}$$

$$= \underline{2\sqrt{6}\text{ units}}$$

9 A function f is defined by $f(x) = e^x \cos x$, for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(a) Using calculus, find the stationary point of $f(x)$ and determine its nature. [5]

$y = e^x \cos x$

$\frac{dy}{dx} = e^x(-\sin x) + \cos x\,(e^x)$

$\quad = e^x(\cos x - \sin x)$

Let $\frac{dy}{dx} = 0$, $\therefore \cos x - \sin x = 0 \quad (\because e^x > 0)$

$\tan x = 1$

$x = \frac{\pi}{4} \quad (\text{given } 0 \leqslant x \leqslant \frac{\pi}{2})$

When $x = \frac{\pi}{4}$, $y = e^{\frac{\pi}{4}}\left(\frac{\sqrt{2}}{2}\right)$

x	$\frac{\pi}{4}^-$	$\frac{\pi}{4}$	$\frac{\pi}{4}^+$
$\frac{dy}{dx}$	/	—	\

Using 1st derivative test,
$\left(\frac{\pi}{4}, \frac{\sqrt{2}e^{\frac{\pi}{4}}}{2}\right)$ is a maximum point.

(b) Integrate by parts twice to show that

$$\int e^{2x} \cos 2x \, dx = \tfrac{1}{4}e^{2x}(\sin 2x + \cos 2x) + c.$$ [4]

Let $u = \cos 2x$ $\quad$ Let $dv = e^{2x}$

$\frac{du}{dx} = -2\sin 2x$ $\quad$ $v = \frac{1}{2}e^{2x}$

Let $u_1 = \sin 2x$ $\quad$ Let $dv_1 = e^{2x}$

$\frac{du_1}{dx} = 2\cos 2x$ $\quad$ $v_1 = \frac{1}{2}e^{2x}$

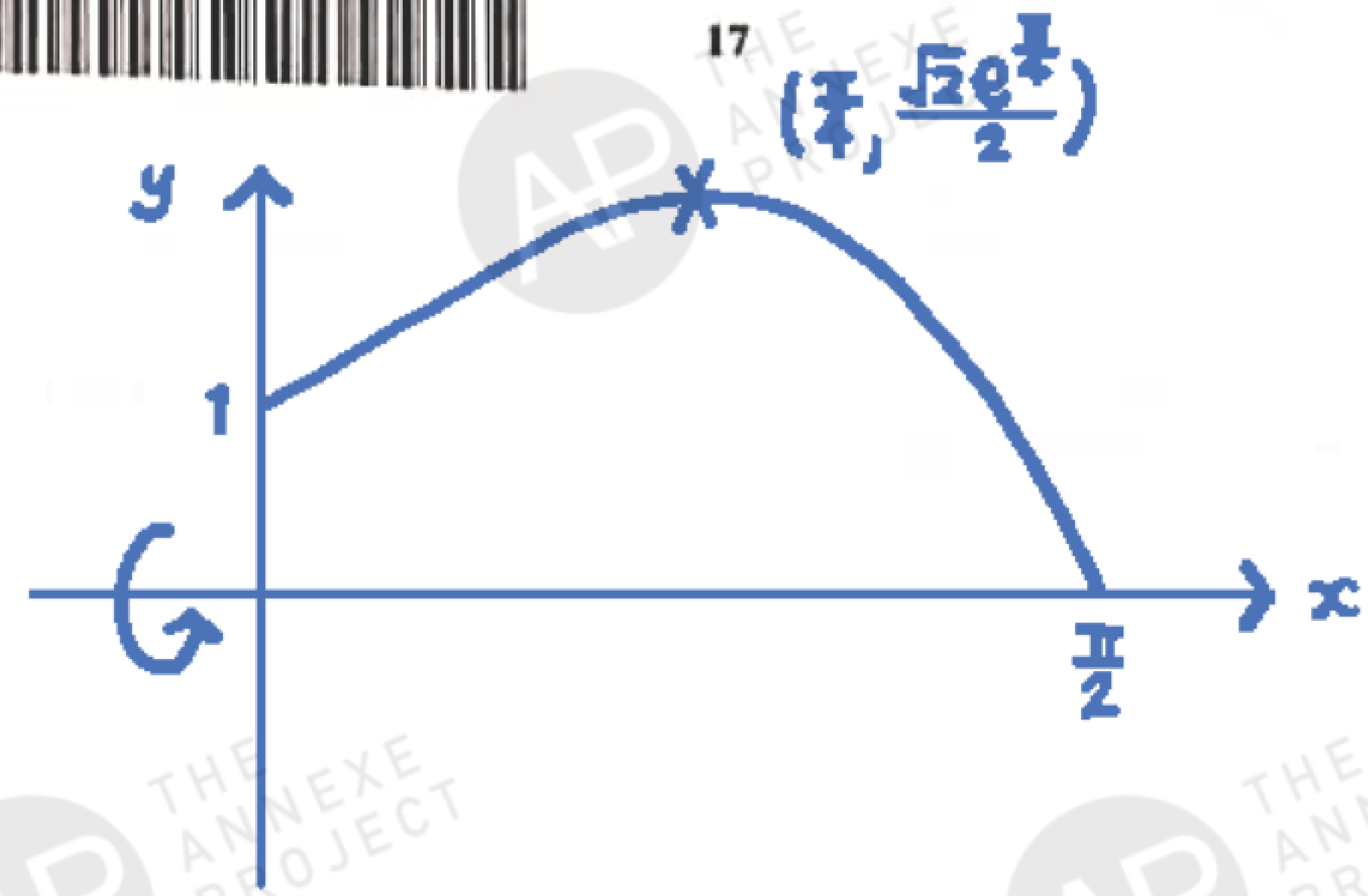
$\int e^{2x} \cos 2x \, dx = \frac{1}{2}e^{2x}\cos 2x + \int e^{2x} \sin 2x \, dx$

$\quad = \frac{1}{2}e^{2x}\cos 2x + \left[\frac{1}{2}e^{2x}\sin 2x - \int e^{2x}\cos 2x \, dx\right]$

$\therefore 2\int e^{2x}\cos 2x \, dx = \frac{1}{2}e^{2x}\cos 2x + \frac{1}{2}e^{2x}\sin 2x$

$\int e^{2x}\cos 2x \, dx = \frac{1}{4}e^{2x}\cos 2x + \frac{1}{4}e^{2x}\sin 2x + C$

$\quad = \underline{\frac{1}{4}e^{2x}(\sin 2x + \cos 2x) + C}$ (shown).



(c) The graph of $y = \mathrm{f}(x)$ is rotated completely about the x-axis. Find the exact volume generated. [4]

$$\text{Vol.} = \pi\int_0^{\frac{\pi}{2}} e^{2x}\cos^2 x \, dx$$

$$= \pi\int_0^{\frac{\pi}{2}} e^{2x}\left[\frac{1+\cos 2x}{2}\right] dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} e^{2x}\,dx + \frac{\pi}{2}\int_0^{\frac{\pi}{2}} e^{2x}\cos 2x \, dx$$

$$= \frac{\pi}{2}\left[\frac{e^{2x}}{2}\right]_0^{\frac{\pi}{2}} + \frac{\pi}{2}\left[\frac{1}{4}e^{2x}(\sin 2x + \cos 2x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}(e^{\pi}-1) + \frac{\pi}{8}(-e^{\pi}-1)$$

$$= \frac{\pi}{8}[2e^{\pi}-2-e^{\pi}-1]$$

$$= \underline{\frac{\pi}{8}(e^{\pi}-3) \text{ units}^3}.$$

10 Scientists model the number of bacteria, N, present at a time t minutes after setting up an experiment. The model assumes that, at any time t, the growth rate in the number of bacteria is kN, for some positive constant k. Initially there are 100 bacteria and it is found that there are 300 bacteria at time $t = 2$.

(a) Write down and solve a differential equation involving N, t and k. Find k and the time it takes for the number of bacteria to reach 1000. [4]

$$\frac{dN}{dt} = kN$$

$$\int \frac{1}{N}\,dN = \int k\,dt$$

$$\ln N = kt + C$$

When $t = 0$, $N = 100$: $\ln 100 = C$

When $t = 2$, $N = 300$: $\ln 300 = 2k + \ln 100$

$$2k = \ln 3$$

$$k = \tfrac{1}{2}\ln 3$$

When $N = 1000$: $\ln 1000 = (\tfrac{1}{2}\ln 3)t + \ln 100$

$$(\tfrac{1}{2}\ln 3)t = \ln 10$$

$$t = 4.19 \text{ mins}$$

The scientists repeat the experiment, again with an initial number of 100 bacteria. The growth rate, kN, for the number of bacteria is the same as that found in part (a). This time they add an anti-bacterial solution which they model as reducing the number of bacteria by d bacteria per minute.

(b) Write down and solve a differential equation, giving t in terms of N and d. Hence find N in terms of t and d. [5]

$$\frac{dN}{dt} = (\tfrac{1}{2}\ln 3)N - d$$

$$\int \frac{1}{(\frac{1}{2}\ln 3)N - d}\,dN = \int dt$$

$$\frac{1}{\frac{1}{2}\ln 3}\int \frac{(\frac{1}{2}\ln 3)}{(\frac{1}{2}\ln 3)N - d}\,dN = t + C$$

$$\frac{2}{\ln 3}\ln\left|(\tfrac{1}{2}\ln 3)N - d\right| = t + C$$

When $t = 0$, $N = 100$: $C = \frac{2}{\ln 3}\ln\left|50\ln 3 - d\right|$

$$\therefore t = \frac{2}{\ln 3}\ln\left|(\tfrac{1}{2}\ln 3)N - d\right| - \frac{2}{\ln 3}\ln\left|50\ln 3 - d\right|$$

$$= \frac{2}{\ln 3}\ln\left|\frac{(\frac{1}{2}\ln 3)N - d}{50\ln 3 - d}\right|$$

$\frac{t\ln 3}{2} = \ln\left|\frac{N\ln 3 - 2d}{100\ln 3 - 2d}\right|$

$\left|\frac{N\ln 3 - 2d}{100\ln 3 - 2d}\right| = e^{\frac{t\ln 3}{2}}$

$\frac{N\ln 3 - 2d}{100\ln 3 - 2d} = \pm e^{\frac{t\ln 3}{2}}$

Given $N = 100$ when $t = 0$:

we reject $-e^{\frac{t\ln 3}{2}}$,

$\therefore \frac{N\ln 3 - 2d}{100\ln 3 - 2d} = e^{\frac{t\ln 3}{2}}$

$N\ln 3 - 2d = e^{\frac{t\ln 3}{2}}[100\ln 3 - 2d]$

$N\ln 3 = e^{\frac{t\ln 3}{2}}(100\ln 3 - 2d) + 2d$

$N = \frac{e^{\frac{t\ln 3}{2}}(100\ln 3 - 2d) + 2d}{\ln 3}$

(c) (i) Find the range of values of d for which the number of bacteria will decrease. [1]

since $e^{\frac{t\ln 3}{2}} > 0$, $100\ln 3 - 2d < 0$

$2d > 100\ln 3$

$d > 50\ln 3$

(ii) In the case where $d = 58$, find the time taken for the number of bacteria to reach zero. [2]

$N = \frac{e^{\frac{t\ln 3}{2}}(100\ln 3 - 2d) + 2d}{\ln 3}$

$0 = e^{\frac{t\ln 3}{2}}(100\ln 3 - 116) + 116$

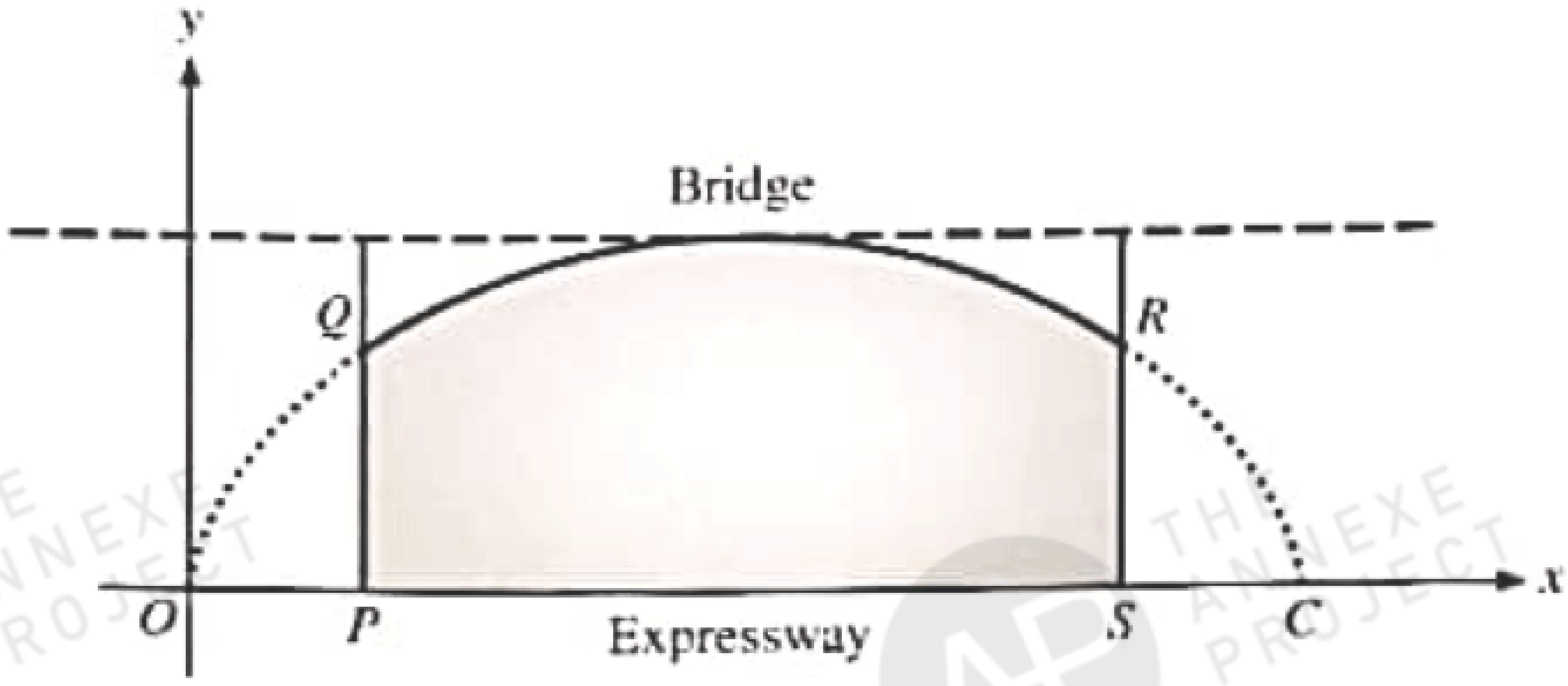
$e^{\frac{t\ln 3}{2}}(100\ln 3 - 116) = -116$

$e^{\frac{t\ln 3}{2}} = 18.896$

$\frac{t\ln 3}{2} = 2.93896661$

$\therefore t = 5.35$ mins

11 Civil engineers design bridges to span over expressways. The diagram below represents a bridg an expressway, PS.



In the diagram, PQ and SR are parallel to the y-axis, and $PQ = SR$. The arch of the bridge, QR, i part of the curve $OQRC$ with parametric equations

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta), \qquad \text{for } 0 \leqslant \theta \leqslant 2\pi,$$

where a is a positive constant. The units of x and y are metres.

At the point Q, $\theta = \beta$ and at the point R, $\theta = 2\pi - \beta$.

(a) Find, in terms of a and β, the distance PS.

x-coordinate of P = x-coordinate of Q
$= a(\beta - \sin\beta)$

x-coordinate of S = x-coordinate of R
$= a[2\pi - \beta - \sin(2\pi - \beta)]$
$= a[(2\pi - \beta) - (-\sin\beta)]$
$= a(2\pi - \beta + \sin\beta)$

4th quadrant
$\sin(360° - \theta) = -\sin\theta$

Length of PS $= a(2\pi - \beta + \sin\beta) - a(\beta - \sin\beta)$
$= 2a\pi - 2a\beta + 2a\sin\beta$
$= \underline{2a(\pi - \beta + \sin\beta)}$

(b) Show that the area of the shaded region on the diagram, representing the area under the bridge, is

$$\tfrac{1}{2}a^2(6\pi - 6\beta + 8\sin\beta - \sin 2\beta).$$ [6]

$x = a(\theta - \sin\theta)$

$\therefore \dfrac{dx}{d\theta} = a - a\cos\theta = a(1-\cos\theta)$

$\text{Area} = \int_{x_p}^{x_s} y \, dx$

$= \int_{\beta}^{2\pi-\beta} a(1-\cos\theta)\cdot a(1-\cos\theta)\, d\theta$

$= a^2\int_{\beta}^{2\pi-\beta} 1 - 2\cos\theta + \cos^2\theta \; d\theta$

Double Angle
$\cos 2\theta = 2\cos^2\theta - 1$
$\therefore \cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

$= a^2\int_{\beta}^{2\pi-\beta} 1 - 2\cos\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}\; d\theta$

$= a^2\int_{\beta}^{2\pi-\beta} \tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta \; d\theta$

$= a^2\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_{\beta}^{2\pi-\beta}$

$= a^2\left[\tfrac{3}{2}(2\pi-\beta) - 2(-\sin\beta) + \tfrac{1}{4}(-\sin 2\beta) - \tfrac{3}{2}\beta + 2\sin\beta - \tfrac{1}{4}\sin 2\beta\right]$

$= a^2\left[3\pi - 3\beta + 4\sin\beta - \tfrac{1}{2}\sin 2\beta\right]$

$= \underline{\tfrac{1}{2}a^2(6\pi - 6\beta + 8\sin\beta - \sin 2\beta)}$ (shown).

(c) It is given that the area under the bridge, in square metres, is $7.8159a^2$. Find the value of β. [1]

$3\pi - 3\beta + 4\sin\beta - \tfrac{1}{2}\sin 2\beta = 7.8159$

By GC: $\beta = 1.9000021$
$= \underline{1.90}$

11 [Continued]

(d) The width of the expressway, PS, is 50 metres. Find the greatest and least heights of the arch, QR, above the expressway. [4]

From part (a): $PS = 2a(\pi + \sin 1.90 - 1.90)$

$50 = 2a(\pi + \sin 1.90 - 1.90)$

$\therefore a = \underline{11.4265}$

Since $y = a(1-\cos\theta)$

$= 11.4265(1-\cos\theta)$

Greatest $y = 11.4265(1-\cos\pi)$

$= \underline{22.9 \text{ m}}$

Least $y = 11.4265(1-\cos\beta)$

$= 11.4265(1-\cos 1.90)$

$= \underline{15.1 \text{ m}}$