The suggested solution are prepared by Mr Alvin Yeo. Mr Yeo will hold no liability for any errors.

MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE
General Certificate of Education Advanced Level
Higher 2

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**MATHEMATICS** **9758/01**

Paper 1 For Examination from 2017

SPECIMEN PAPER 3 hours

1 A circular ink-blot is expanding such that the rate of change of its diameter $D$ with respect to time $t$ is 0.25 cm/s. Find the rate of change of both the circumference and the area of the circle with respect to $t$ when the radius of the circle is 1.5 cm. Give your answers correct to 4 decimal places. [4]

Given $\frac{dD}{dt} = 0.25$ cm/s

Find $\frac{dC}{dt}$ and $\frac{dA}{dt}$ when $r = 1.5$ cm.

Missing: $\frac{dC}{dt} = \frac{dD}{dt} \times \frac{dC}{dD}$

Since $C = \pi D$

$\frac{dC}{dD} = \pi$

$= 0.25 \times \pi$

$= \underline{0.7854 \text{ cm/s}}$

Missing: $\frac{dA}{dt} = \frac{dD}{dt} \times \frac{dA}{dD}$

Since $A = \pi r^2$

$= \pi\left(\frac{D}{2}\right)^2$

$= \frac{\pi}{4}D^2$

$\frac{dA}{dD} = \frac{\pi}{2}D$

$= 0.25 \times \frac{\pi}{2}(1.5 \times 2)$

$= \underline{1.1781 \text{ cm}^2\text{/s}}$

2 The curve $C$ with equation $y = x^3$ is transformed onto the curve with equation $y = f(x)$ by a translation of 2 units in the negative $x$-direction, followed by a stretch of factor $\frac{1}{2}$ parallel to the $y$-axis, followed by a translation of 1 unit in the positive $y$-direction.

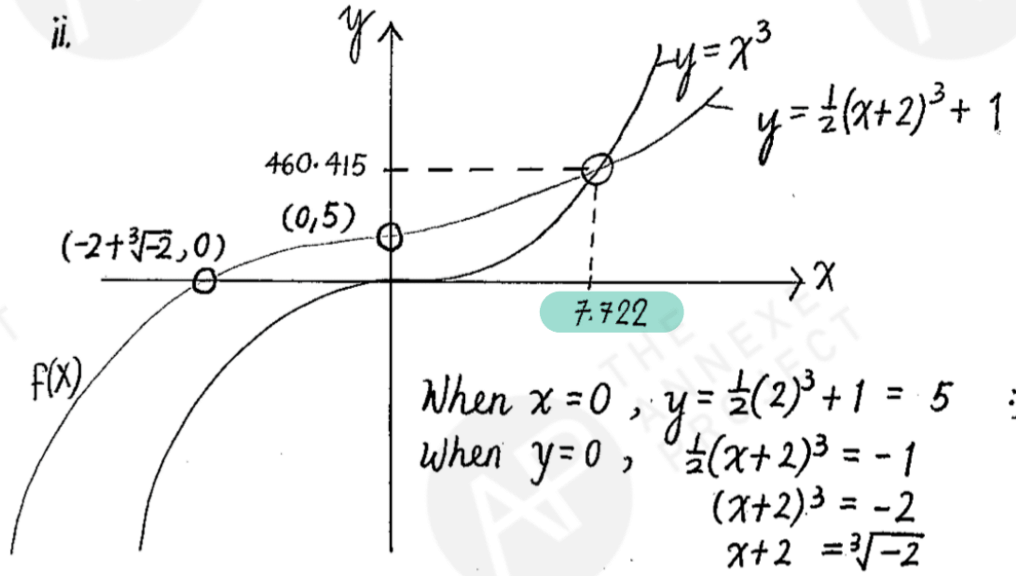
(i) Write down the equation of the new curve. [1]

(ii) Sketch $C$ and the curve with equation $y = f(x)$ on the same diagram, stating the exact values of the coordinates of the points where $y = f(x)$ crosses the $x$- and $y$-axes. Find the $x$-coordinate(s) of the points(s) where the two curves intersect, giving your answer(s) correct to 3 decimal places. [4]

i

$$y = x^3 \longrightarrow y = (x+2)^3 \longrightarrow y = \tfrac{1}{2}(x+2)^3 \longrightarrow \underline{y = \tfrac{1}{2}(x+2)^3 + 1}$$

ii.




When $x = 0$, $y = \frac{1}{2}(2)^3 + 1 = 5$ : $\underline{(0,5)}$

When $y = 0$, $\frac{1}{2}(x+2)^3 = -1$

$(x+2)^3 = -2$

$x+2 = \sqrt[3]{-2}$

$x = -2 + \sqrt[3]{-2}$ : $\underline{(-2+\sqrt[3]{-2}, 0)}$

3 (i) Sketch the curve with equation $y = \frac{x^2-12}{x}$, giving the exact coordinates of the point(s) where the curve crosses the axes and the equations of any asymptotes. [4]

(ii) Hence, or otherwise, solve the inequality $\frac{x^2-12}{x} < 1$. [3]

i. $y = \frac{x^2-12}{x}$

$= \frac{(x-\sqrt{12})(x+\sqrt{12})}{x}$

When $y=0$, $x^2 = 12$

$x = \pm\sqrt{12}$

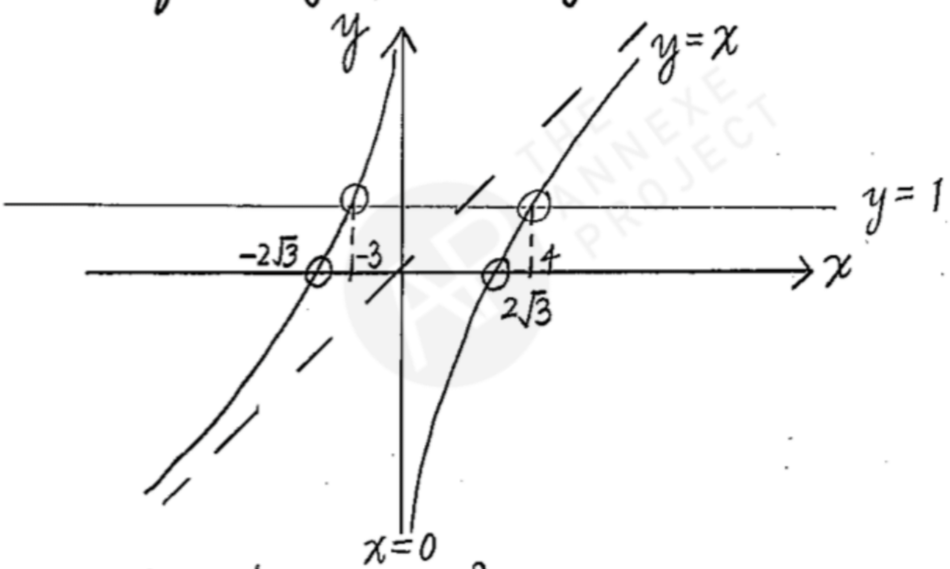
$= \pm 2\sqrt{3}$

Vertical Asymptote : $x = 0$

$y = \frac{x^2-12}{x} = x - \frac{12}{x}$

When $x \to \pm\infty$, $y \to x$ since $\frac{12}{x} \to 0$

Oblique Asymptote : $y = x$



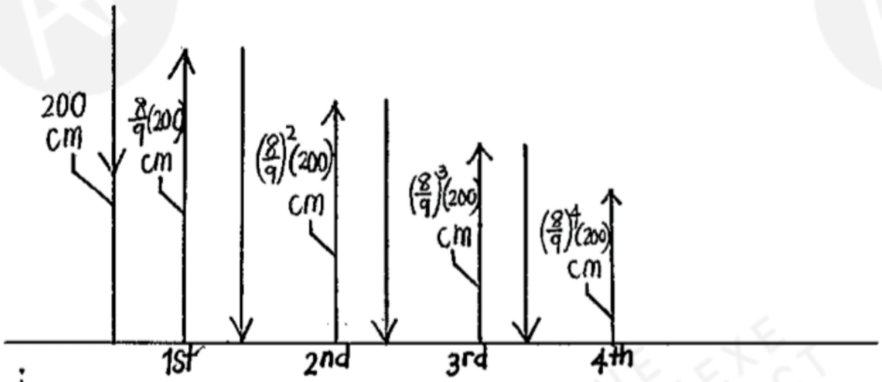

ii. From the above graph,

for $\frac{x^2-12}{x} < 1$, $0 < x < 4$ or $x < -3$

4 A science student is investigating the elasticity of a new compound. She drops a ball made of the new compound vertically onto a hard surface and measures the height reached by the ball after each successive bounce. She drops the ball form an initial height of 200 cm and she estimates that the height the ball reaches after each bounce is $\frac{8}{9}$ of the height reached by the previous bounce.

(i) Find the total distance that the ball has travelled when it reaches the highest point after the fourth bounce. Give your answer correct to the nearest centimeter. [2]

(ii) The ball is considered to have stopped bouncing when a bounce first results in the height the ball reaches being less than 0.01 cm. Find how many bounces the ball has made and the total distance that the ball has travelled in this case. Give your answer correct to the nearest centimetre. [6]




i. total distance $= 200 + 2[\frac{8}{9}(200)] + 2[(\frac{8}{9})^2(200)] + 2[(\frac{8}{9})^3(200)] + (\frac{8}{9})^4(200)$

$= 1277.40 = 1277 \text{ cm}$

ii. Let $T_n$ be $n^{th}$ bounce of a ball:

$T_n = (\frac{8}{9})^n(200) < 0.01$

$(\frac{8}{9})^n < 0.00005$

$n > 84.1$

$\therefore T_n < 0.01$ cm at the $85^{th}$ bounce.

Total distance $= 200 + 2[\text{Sum of 84 bounces}]$

$= 200 + 2\left[\dfrac{\frac{8}{9}(200)[1-(\frac{8}{9})^{84}]}{1-(\frac{8}{9})}\right]$

$= 200 + 3199.838$

$= 3399.84$

$= 3400 \text{ cm}$

5 The curve $C$ has equation $y = \frac{1}{x}(\ln x)^3$, where $x > 1$.

(i) Find the exact $x$-coordinate, $x = x_1$m if the turning point on $C$ and explain whether it is a maximum or a minimum turning point. [4]

(ii) Without using a calculator, find the exact area of the region between $C$, the $x$-axis and the lines with equations $x = e$ and $x = x_1$. [3]

i. $y = \frac{1}{x}(\ln x)^3, \quad x > 1$

$\frac{dy}{dx} = \frac{1}{x}(3)(\ln x)^2\left(\frac{1}{x}\right) + (\ln x)^3\left(\frac{-1}{x^2}\right)$

$= \frac{(\ln x)^2}{x^2}[3 - \ln x]$

let $\frac{dy}{dx} = 0$, $\therefore (\ln x)^2(3 - \ln x) = 0$

i.e. $\ln x = 0$ or $\ln x = 3$

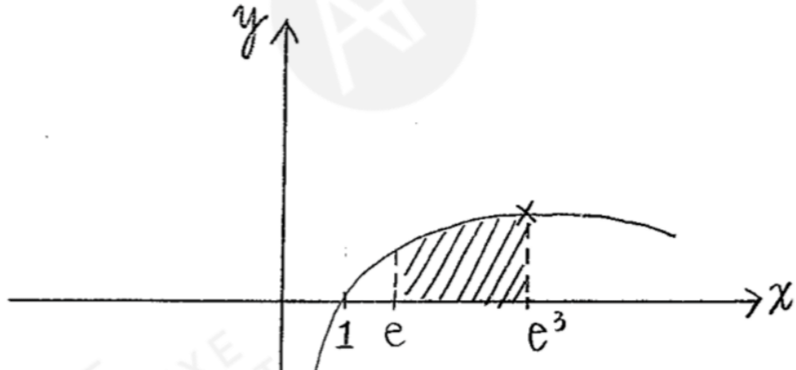
$x = 1$ or $x = e^3$

(Rej) $\because x > 1$

| $x$ | $(e^3)^-$ | $e^3$ | $(e^3)^+$ |
|---|---|---|---|
| $\frac{dy}{dx}$ | / | — | \ |

At $x = e^3$, it is a maximum point.

ii.




Let $u = (\ln x)^3$

$\frac{du}{dx} = 3(\ln x)^2\left(\frac{1}{x}\right)$

Let $dv = \frac{1}{x}$

$v = \ln x$

$A = \int \frac{1}{x}(\ln x)^3\,dx$

$= (\ln x)^4 - \int (\ln x)\cdot 3(\ln x)^2\left(\frac{1}{x}\right)dx$

$= (\ln x)^4 - 3\int \frac{1}{x}(\ln x)^3\,dx$

Since $\int \frac{1}{x}(\ln x)^3\,dx = (\ln x)^4 - 3\int \frac{1}{x}(\ln x)^3\,dx$

then $4\int \frac{1}{x}(\ln x)^3\,dx = (\ln x)^4$

$\int \frac{1}{x}(\ln x)^3\,dx = \frac{1}{4}(\ln x)^4 + C$

$\therefore \int_e^{e^3} \frac{1}{x}(\ln x)^3\,dx = \frac{1}{4}\left[(\ln x)^4\right]_e^{e^3}$

$= \frac{1}{4}\left[(\ln e^3)^4 - (\ln e)^4\right]$

$= \frac{1}{4}[81 - 1] = 20 \text{ units}^2$

6 (a) The non-zero vectors **a, b** and **c** are such that **a** × **b** = **c** × **a**. Given that **b**≠ − **c**, find the linear relationship between **a, b** and **c**. [3]

(b) The variable vector **v** satisfies the equation $\mathbf{v}\times(\mathbf{i}-3\mathbf{k})=2\mathbf{j}$. Find the set of vectors **v** and describe this set geometrically. [5]

a. Given $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{c} \times \underset{\sim}{a}$

$(\underset{\sim}{a} \times \underset{\sim}{b}) - (\underset{\sim}{c} \times \underset{\sim}{a}) = 0$

$(\underset{\sim}{a} \times \underset{\sim}{b}) + (\underset{\sim}{a} \times \underset{\sim}{c}) = 0$

$\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = 0$

$\underset{\sim}{a}$ is parallel to $\underset{\sim}{b} + \underset{\sim}{c}$

Hence, $\underset{\sim}{a} = k(\underset{\sim}{b} + \underset{\sim}{c})$, $k \in \mathbb{R}$

b. Let $\underset{\sim}{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -3b \\ -(-3a-c) \\ -b \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

Hence, $\underset{\sim}{b} = 0$, $3\underset{\sim}{a} + \underset{\sim}{c} = 2$

$\therefore 3\underset{\sim}{a} = 2 - \underset{\sim}{c}$

$\underset{\sim}{a} = \frac{2}{3} - \frac{1}{3}\underset{\sim}{c}$

let $\underset{\sim}{c}$ be a parameter $\lambda$

then $\underset{\sim}{a} = \frac{2}{3} - \frac{1}{3}\lambda$

$\underset{\sim}{b} = 0$

$\underset{\sim}{c} = \lambda$

$\Rightarrow \underset{\sim}{v} = \begin{pmatrix} \frac{2}{3} - \frac{1}{3}\lambda \\ 0 \\ \lambda \end{pmatrix} = \begin{pmatrix} 2/3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2/3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

$\underset{\sim}{v}$ is a line with equation

$l: \underset{\sim}{r} = \begin{pmatrix} 2/3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

7 **Do not use a calculator in answering this question.**

(a) Showing your working, find the complex numbers z and $w$ which satisfy the simultaneous Equations

$$2\mathrm{i}z + (1 - 2\mathrm{i})w = 4 \text{ and}$$
$$(1+\mathrm{i})z + (2+\mathrm{i})w = 3. \qquad [6]$$

(ii) The complex number $u$ is given by $u = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \pi$. Show that $1 - u^2 = -2\mathrm{i}u\sin\theta$ and hence or otherwise find the modulus and argument of $1 - u^2$ in terms of $\theta$. [5]

a. Given $2iz + (1-2i)w = 4$ ) multiply both sides by $i$

$2i^2z + (i - 2i^2)w = 4i$

$-2z + iw + 2w = 4i$

$2z = 2w + iw - 4i$

$z = w + \frac{i}{2}w - 2i$ ——— ①

Sub ① into the other given eqn : $(1+i)z + (2+i)w = 3$

$\therefore (1+i)(w + \frac{i}{2}w - 2i) + (2+i)w = 3$

$w + \frac{i}{2}w - 2i + iw - \frac{1}{2}w + 2 + 2w + iw = 3$

$\frac{5}{2}w + \frac{5}{2}iw = 1 + 2i$

$5w + 5iw = 2 + 4i$

$w = \frac{2+4i}{5+5i} \times \frac{5-5i}{5-5i} = \frac{10+10i+20}{25+25} = \frac{3}{5} + \frac{1}{5}i$

$= \frac{1}{5}(3+i)$

$\therefore z = \frac{3}{5} + \frac{1}{5}i + \frac{i}{2}(\frac{3}{5} + \frac{1}{5}i) - 2i$

$= \frac{3}{5} + \frac{1}{5}i + \frac{3}{10}i - \frac{1}{10} - 2i$

$= \frac{1}{2} - \frac{3}{2}i = \frac{1}{2}(1-3i)$

b. To show $1 - u^2 = -2iu\sin\theta$ , given $u = \cos\theta + i\sin\theta$

LHS $= 1 - u^2$

$= 1 - [\cos\theta + i\sin\theta]^2$

$= 1 - [\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta]$

$= (\sin^2\theta + \cancel{\cos^2\theta}) - \cancel{\cos^2\theta} - 2i\sin\theta\cos\theta + \sin^2\theta$

$= 2\sin^2\theta - 2i\sin\theta\cos\theta$

$= -2i\sin\theta[\cos\theta - \frac{\sin\theta}{i}]$

$= -2i\sin\theta[\cos\theta - \frac{i\sin\theta}{i^2}]$

$= -2i\sin\theta[\cos\theta + i\sin\theta]$

$= -2i\sin\theta(u)$

$=$ RHS (shown)

$|1-u^2| = |-2i\sin\theta(u)|$

$= |-2i||\sin\theta||u|$

$= 2\sin\theta\,|u|$ since $0 < \theta < \pi$

$= 2\sin\theta$ since $|u| = \sqrt{\cos^2\theta + \sin^2\theta}$

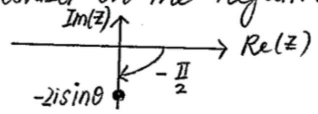
$= 1$

$$\arg(1-u^2) = \arg(-2iu\sin\theta)$$

$$= \arg(-2i\sin\theta) + \arg u$$

$$= -\frac{\pi}{2} + \theta$$

Since $0 < \theta < \pi$
then $\sin\theta$ is a positive number,
hence $-2i(\sin\theta)$ is a negative
number on the negative y-axis,



8 The asteroid, a curve $C$ which is used to characterize various properties of energy and magnetism, has parametric equations

$$x = a\cos^3 t\,, y = a\sin^3 t,$$

where $0 \le t \le \frac{1}{2}\pi$ and $a$ is a positive constant.

(i) Find the equation of the tangent to $C$ at the point $P$ with parameter $p$. [3]

(ii) The tangent at $P$ meets the $x$-axis at the point $A$ and meets the $y$-axis at the point $B$. Show that the length $AB$ depends only on $a$. [3]

It is given that $a = 1$.

(iii) Find a cartesian equation of $C$. [2]

(iv) The region bounded by $C$ and the $x$- and $y$-axes is rotated through $360°$ about the $y$-axis. Find the exact value of the volume of revolution of the solid formed. [4]

i $x = a\cos^3 t$ ; $y = a\sin^3 t$

$\frac{dx}{dt} = a\cdot 3\cos^2 t(-\sin t)$ ; $\frac{dy}{dt} = a\cdot 3\sin^2 t\cos t$

$= -3a\sin t\cos^2 t$ $\quad = 3a\sin^2 t\cos t$

$\therefore \frac{dy}{dx} = \frac{\cancel{3a}\sin^{\cancel{2}} t\,\cancel{\cos t}}{-\cancel{3a}\,\cancel{\sin t}\cos^{\cancel{2}} t} = -\frac{\sin t}{\cos t} = -\tan t$

When $t = p$, $P = (a\cos^3 p\,, a\sin^3 p)$

and $\frac{dy}{dx} = -\tan p$

Equation of tangent:

$y - a\sin^3 p = -\tan p\,(x - a\cos^3 p)$

$y = a\sin^3 p - \frac{\sin p}{\cos p}(x - a\cos^3 p)$

$= a\sin^3 p - (\tan p)x + a\sin p\cos^2 p$

$= a\sin^3 p - x\tan p + a\sin p\,(1-\sin^2 p)$

$= a\sin^3 p - x\tan p + a\sin p - a\sin^3 p$

$y = a\sin p - x\tan p$

Length of 2 points $= \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$ where $A = (x_1, y_1)$, $B = (x_2, y_2)$

ii. Let $y = 0$:

$\therefore x\tan p = a\sin p$

$x = \frac{a\sin p}{\tan p}$

$= \frac{a\sin p}{\left(\frac{\sin p}{\cos p}\right)} = a\cos p$

$\therefore A = (a\cos p\,, 0)$

Let $x = 0$:

$\therefore y = a\sin p$

$B = (0, a\sin p)$

$AB = \sqrt{(a\sin p - 0)^2 + (0 - a\cos p)^2}$

$= \sqrt{a^2\sin^2 p + a^2\cos^2 p}$

$= \sqrt{a^2} = a$ (shown).

Given $a = 1$

(iii). $x = \cos^3 t$ and $y = \sin^3 t$

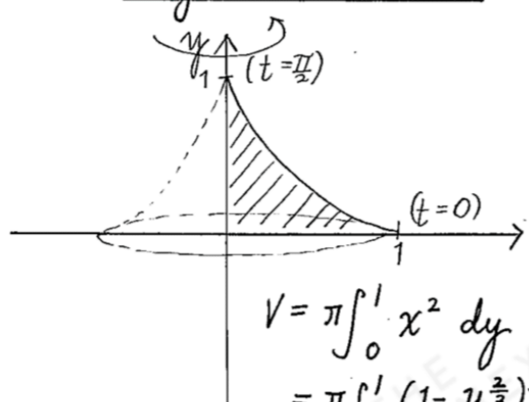
$\sqrt[3]{x} = \cos t \qquad \sqrt[3]{y} = \sin t$

Using the identity $\sin^2\theta + \cos^2\theta = 1$

$$\therefore (\sqrt[3]{y})^2 + (\sqrt[3]{x})^2 = 1$$

$$y^{\frac{2}{3}} + x^{\frac{2}{3}} = 1$$

(iv).




$\because x^{\frac{2}{3}} = 1 - y^{\frac{2}{3}}$

$x^2 = (1 - y^{\frac{2}{3}})^3$

Binomial Theorem:
$(1+x)^3 = 1 + \binom{3}{1}x + \binom{3}{2}x^2 + x^3$

$$V = \pi \int_0^1 x^2 \, dy$$

$$= \pi \int_0^1 (1 - y^{\frac{2}{3}})^3 \, dy$$

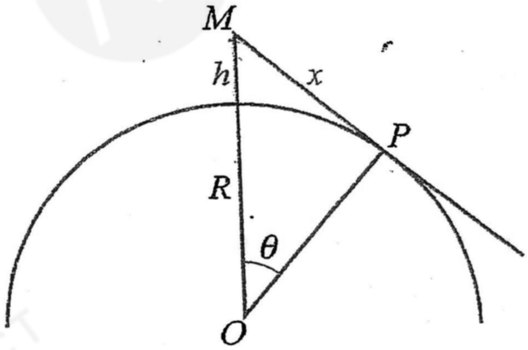
$$= \pi \int_0^1 1 + 3(-y^{\frac{2}{3}}) + 3(-y^{\frac{2}{3}})^2 + (-y^{\frac{2}{3}})^3 \, dy$$

$$= \pi \int_0^1 1 - 3y^{\frac{2}{3}} + 3y^{\frac{4}{3}} - y^2 \, dy$$

$$= \pi \left[ y - \frac{3y^{\frac{5}{3}}}{\frac{5}{3}} + \frac{3y^{\frac{7}{3}}}{\frac{7}{3}} - \frac{y^3}{3} \right]_0^1$$

$$= \pi \left[ y - \frac{9}{5}y^{\frac{5}{3}} + \frac{9}{7}y^{\frac{7}{3}} - \frac{1}{3}y^3 \right]_0^1$$

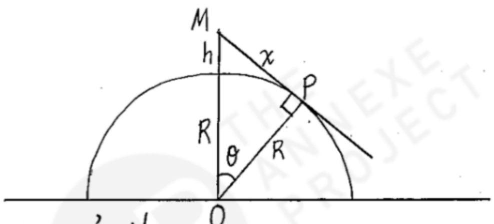
$$= \pi \left[ \frac{16}{105} - 0 \right] = \frac{16\pi}{105} \text{ units}^3.$$




A man $M$ is at the top of a mountain which is of height $h$km. The radius of the earth is assumed to be a constant $R$km. The furthest point on the earth's surface that the man can see is a point $P$ such that $MP = x$km and the angle $POM = \theta$, where $O$ is the centre of the earth (see diagram). You may assume that the height of the man is negligible.

(i) Show that $x = (2hR)^{\frac{1}{2}}(1+\frac{h}{2R})^{\frac{1}{2}}$. [3]

(ii) It is given that $h$ is small compared to $R$. Show that, if $\alpha = \frac{h}{R}$, $\sin\theta \approx (2\alpha)^{\frac{1}{2}}(1-\frac{3}{4}\alpha)$. [5]




i). Using Pythagoras' theorem:

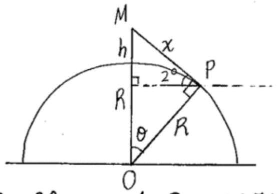
$$(h+R)^2 = x^2 + R^2$$
$$x^2 = (h^2 + 2hR + R^2) - R^2$$
$$= h^2 + 2hR$$
$$= 2hR\left(1 + \frac{h^2}{2hR}\right)$$
$$= 2hR\left(1 + \frac{h}{2R}\right)$$
$$\therefore x = (2hR)^{\frac{1}{2}}\left(1+\frac{h}{2R}\right)^{\frac{1}{2}}$$ (The negative sign is omitted since $x > 0$)

(shown)

ii). 
$$\sin\theta = \frac{x}{h+R}$$
$$= \frac{(2hR)^{\frac{1}{2}}(1+\frac{h}{2R})^{\frac{1}{2}}}{h+R}$$
$$= \frac{\frac{(2hR)^{\frac{1}{2}}}{R}(1+\frac{h}{2R})^{\frac{1}{2}}}{\frac{h}{R}+1}$$
divide both numerator and denominator by $R$.
$$= \frac{\left(\frac{2h}{R}\right)^{\frac{1}{2}}\left(1+\frac{h}{2R}\right)^{\frac{1}{2}}}{\frac{h}{R}+1}$$
$$= \frac{(2\alpha)^{\frac{1}{2}}(1+\frac{1}{2}\alpha)^{\frac{1}{2}}}{\alpha+1}$$ (since $\alpha = \frac{h}{R}$)
$$= (2\alpha)^{\frac{1}{2}}[1+\tfrac{1}{2}\alpha]^{\frac{1}{2}}[1+\alpha]^{-1}$$
$$= (2\alpha)^{\frac{1}{2}}\left[1+\tfrac{1}{2}(\tfrac{1}{2}\alpha)+\tfrac{\frac{1}{2}(-\frac{1}{2})}{2!}(\tfrac{1}{2}\alpha)^2+\cdots\right]\left[1-\alpha+\tfrac{(-1)(-2)}{2!}\alpha^2+\cdots\right]$$
$$= (2\alpha)^{\frac{1}{2}}[1+\tfrac{1}{4}\alpha+\cdots][1-\alpha+\cdots]$$
$$\approx (2\alpha)^{\frac{1}{2}}(1-\alpha+\tfrac{1}{4}\alpha)$$
$$= (2\alpha)^{\frac{1}{2}}(1-\tfrac{3}{4}\alpha)$$
(shown).

$\boxed{\text{Since } h \text{ is small, } \alpha \text{ is small, hence we neglect } \alpha^2 \text{ and above powers of } \alpha}$

iii).




by similar $\Delta$s , $\theta = 2°$, and $R = 6375\,km$

$\sin 2° \approx (2\alpha)^{\frac{1}{2}}(1 - \frac{3}{4}\alpha)$

Using GC : $\underline{\alpha = 6.0954 \times 10^{-4}}$ or 1.3045 (Rej because $\alpha$ is small)

$h = \alpha R$

$= 6.0954 \times 10^{-4} \times 6375\,km$

$= 3.8858$

$= \underline{3.89\,km}$

10 The point $A$ has coordinates $(-1, 2, -1)$. The line $l$ has equation $\frac{x}{2} = \frac{y+1}{-3} = \frac{z-2}{1}$.

(i) Find the cartesian equation of the plane $\pi$ which contains $A$ and is perpendicular to $l$. [2]

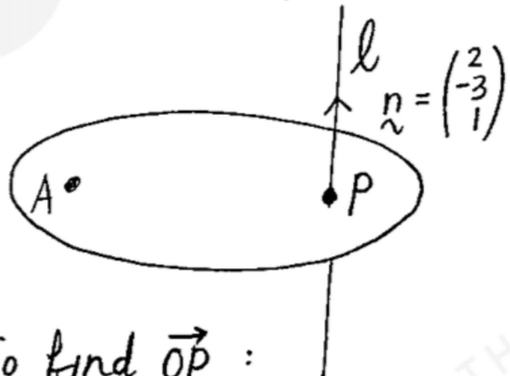
(ii) Hence, or otherwise, find the coordinates of the point $P$ on $l$ which is closest to $A$. [3]

(iii) The line $m$ passes through the point with coordinates $(4, -5, 10)$ and $P$. The line n lies in the same plane as $l$ and $m$. Find a cartesian equation for $n$ if $n$ is the reflection of the line $m$ about the line $l$. [6]

Given $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$l: \underset{\sim}{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$

i).




$\underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$2x - 3y + z = -2 - 6 - 1$

$= -9$

ii). To find $\overrightarrow{OP}$ :

Sub eqn of $l$ into eqn. of $\pi$

$\begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -9$

$4\lambda + 3 + 9\lambda + 2 + \lambda = -9$

$14\lambda = -14$

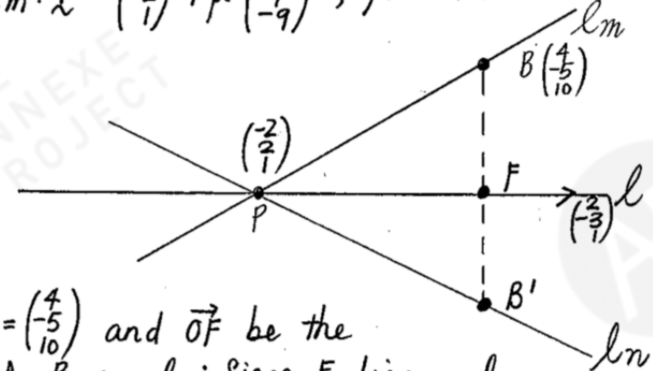
$\lambda = -1$

$\therefore \overrightarrow{OP} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

coordinates of $P = (-2, 2, 1)$

iii). direction of $l_m = \begin{pmatrix}-2\\2\\1\end{pmatrix} - \begin{pmatrix}4\\-5\\10\end{pmatrix} = \begin{pmatrix}-6\\7\\-9\end{pmatrix}$

$\therefore\ l_m : \underset{\sim}{r} = \begin{pmatrix}-2\\2\\1\end{pmatrix} + \mu \begin{pmatrix}-6\\7\\-9\end{pmatrix},\ \mu \in \mathbb{R}$




Note that P lies on both $l_m$ and $l$

Let $\vec{OB} = \begin{pmatrix}4\\-5\\10\end{pmatrix}$ and $\vec{OF}$ be the foot of B on $l$ : Since F lies on $l$,

Step 1: $\vec{OF} = \begin{pmatrix}2\lambda\\-1-3\lambda\\2+\lambda\end{pmatrix}$

2: $\vec{BF} = \begin{pmatrix}2\lambda\\-1-3\lambda\\2+\lambda\end{pmatrix} - \begin{pmatrix}4\\-5\\10\end{pmatrix} = \begin{pmatrix}2\lambda-4\\4-3\lambda\\-8+\lambda\end{pmatrix}$

3: $\vec{BF} \cdot \begin{pmatrix}2\\-3\\1\end{pmatrix} = 0$

i.e. $\begin{pmatrix}2\lambda-4\\4-3\lambda\\-8+\lambda\end{pmatrix} \cdot \begin{pmatrix}2\\-3\\1\end{pmatrix} = 0$

$$4\lambda - 8 - 12 + 9\lambda - 8 + \lambda = 0$$
$$14\lambda = 28$$
$$\lambda = 2$$
$$\therefore\ \vec{OF} = \begin{pmatrix}4\\-7\\4\end{pmatrix}$$

4: By ratio theorem:

$$\vec{OF} = \frac{\vec{OB} + \vec{OB}'}{2}$$

$$\therefore\ \vec{OB}' = 2\vec{OF} - \vec{OB} = \begin{pmatrix}8\\-14\\8\end{pmatrix} - \begin{pmatrix}4\\-5\\10\end{pmatrix} = \begin{pmatrix}4\\-9\\-2\end{pmatrix}$$

$$\vec{PB}' = \begin{pmatrix}4\\-9\\-2\end{pmatrix} - \begin{pmatrix}-2\\2\\1\end{pmatrix} = \begin{pmatrix}6\\-11\\-3\end{pmatrix}$$

$$\therefore\ l_n : \underset{\sim}{r} = \begin{pmatrix}4\\-9\\-2\end{pmatrix} + \alpha \begin{pmatrix}6\\-11\\-3\end{pmatrix},\ \alpha \in \mathbb{R}$$

Cartesian form: $x = 4 + 6\alpha$
$y = -9 - 11\alpha$
$z = -2 - 3\alpha$

$$\Rightarrow \frac{x-4}{6} = \frac{y+9}{-11} = \frac{z+2}{-3}$$

11 A pond has a surface area of $10\mathrm{m}^2$. Biologists have planted an area of new weeds. They estimate how many weeds there are and the rate at which they are spreading by finding the area of the pond the weeds cover at various times. They believe the area, $A\mathrm{m}^2$, of weeds present at time *t* months is such that the rate at which the area is increasing is proportional to the product of the area of pond covered by the weeds a d the area of the pond not covered by the weeds. It is known that the initial area of weeds is $2\mathrm{m}^2$ and that the area of weeds is $4\mathrm{m}^2$ after 5 months.

(i) Wire down a differential equation expressing the relation between *A* and *t*. Find the time at which 80% of the pond is covered in weeds, giving your answer correct to 2 decimal places. [8]

(ii) Given that the experiment is stopped after 2 years, find the area of pond covered by weed, giving your answer correct to 2 decimal places. [2]

(ii) Write the solution of the differential equation in the form $A = f(t)$ and sketch this curve. [4]

Let $A$ be area of weeds (in $m^2$) at any time $t$ (months):

$$\frac{dA}{dt} = kA(10-A)$$

$$\int \frac{1}{A(10-A)}\,dA = k\int dt$$

$$\frac{1}{10}\int \frac{1}{A} + \frac{1}{10-A}\,dA = k\int dt$$

By Partial Fractions:

$$\frac{1}{A(10-A)} = \frac{b}{A} + \frac{c}{(10-A)}$$

$$1 = b(10-A) + cA$$

$$1 = 10b - bA + cA$$

Comparing coefficients,

$10b = 1$ ; $0 = -b + c$

$b = \frac{1}{10}$ ; $b = c$

$c = \frac{1}{10}$

$$\frac{1}{10}\left[\ln|A| - \ln|10-A|\right] = kt + C$$

$$\ln\left|\frac{A}{10-A}\right| = 10kt + 10C$$

$$\left|\frac{A}{10-A}\right| = e^{10kt+10C}$$

$$\frac{A}{10-A} = \pm e^{10C} e^{10kt}$$

$$\frac{A}{10-A} = De^{10kt}, \text{ where } D = \pm e^{10C}$$

$$A = 10De^{10kt} - ADe^{10kt}$$

$$A + ADe^{10kt} = 10De^{10kt}$$

$$A = \frac{10De^{10kt}}{1+De^{10kt}}$$

given $A = 2$ when $t = 0$: $2 = \frac{10D}{1+D}$

$$2 + 2D = 10D$$

$$\therefore 8D = 2$$

$$D = \frac{1}{4}$$

when $t = 5$, $A = 4$: $4 = \frac{10(\frac{1}{4})e^{50k}}{1+\frac{1}{4}e^{50k}}$

$$4 + e^{50k} = \frac{5}{2}e^{50k}$$

$$\frac{3}{2}e^{50k} = 4$$

$$e^{50k} = \frac{8}{3}$$

$$50k = \ln\frac{8}{3}$$

$$k = \frac{1}{50}\ln\frac{8}{3}$$

$$\therefore A = \frac{10(\frac{1}{4})e^{10[\frac{1}{50}\ln\frac{8}{3}]t}}{1+\frac{1}{4}e^{10[\frac{1}{50}\ln\frac{8}{3}]t}}$$

$$= \frac{\frac{5}{2}e^{(\frac{1}{5}\ln\frac{8}{3})t}}{1+\frac{1}{4}e^{(\frac{1}{5}\ln\frac{8}{3})t}}$$

multiply both numerator & denominator by $4e^{-(\frac{1}{5}\ln\frac{8}{3})t}$

$$= \underline{\frac{10}{1+4e^{-(\frac{1}{5}\ln\frac{8}{3})t}}}$$

When $A = 0.8(10)$
$= 8\ m^2$ :

$$8 = \frac{10}{1+4e^{-(\frac{1}{5}\ln\frac{8}{3})t}}$$

$$8 + 32e^{-(\frac{1}{5}\ln\frac{8}{3})t} = 10$$

$$e^{-(\frac{1}{5}\ln\frac{8}{3})t} = \frac{1}{16}$$

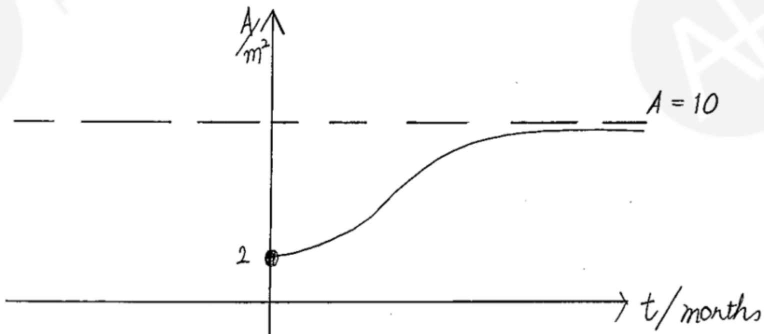
hence, $\left(-\frac{1}{5}\ln\frac{8}{3}\right)t = \ln\frac{1}{16}$

$t = 14.13$ months

ii). When $t = 24$:

$$A = \frac{10}{1+4e^{\left(-\frac{1}{5}\ln\frac{8}{3}\right)(24)}} = 9.6517 = 9.65\ m^2$$

iii). from part ①: $A = \dfrac{10}{1+4e^{-\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}$




As $t \to \infty$, $e^{-\left(\frac{1}{5}\ln\frac{8}{3}\right)t} \to 0$

$\therefore A \longrightarrow 10\ m^2$