MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE
General Certificate of Education Advanced Level
Higher 2

MATHEMATICS

9758/01

Paper 1

October/November 2017

3 hours

Additional Materials: Answer Paper
Graph paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

CAMBRIDGE
International Examinations

1 Using standard series from the List of Formulae (MF26), expand $e^{2x}\ln(1+ax)$ as far as the term in x^3, where a is a non-zero constant. Hence find the value of a for which there is no term in x^2. [4]

$$e^{2x}\ln(1+ax) = \left[1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots\right]\left[ax-\frac{(ax)^2}{2}+\frac{(ax)^3}{3}-\cdots\right]$$
$$= \left[1+2x+2x^2+\tfrac{4}{3}x^3+\cdots\right]\left[ax-\tfrac{1}{2}a^2x^2+\tfrac{1}{3}a^3x^3-\cdots\right]$$
$$= \left[ax-\tfrac{1}{2}a^2x^2+\tfrac{1}{3}a^3x^3+2ax^2-a^2x^3+2ax^3+\cdots\right]$$
$$\approx \underline{ax+(2a-\tfrac{1}{2}a^2)x^2+(2a-a^2+\tfrac{1}{3}a^3)x^3}$$

Let $2a-\frac{1}{2}a^2=0$

$\frac{1}{2}a(4-a)=0$

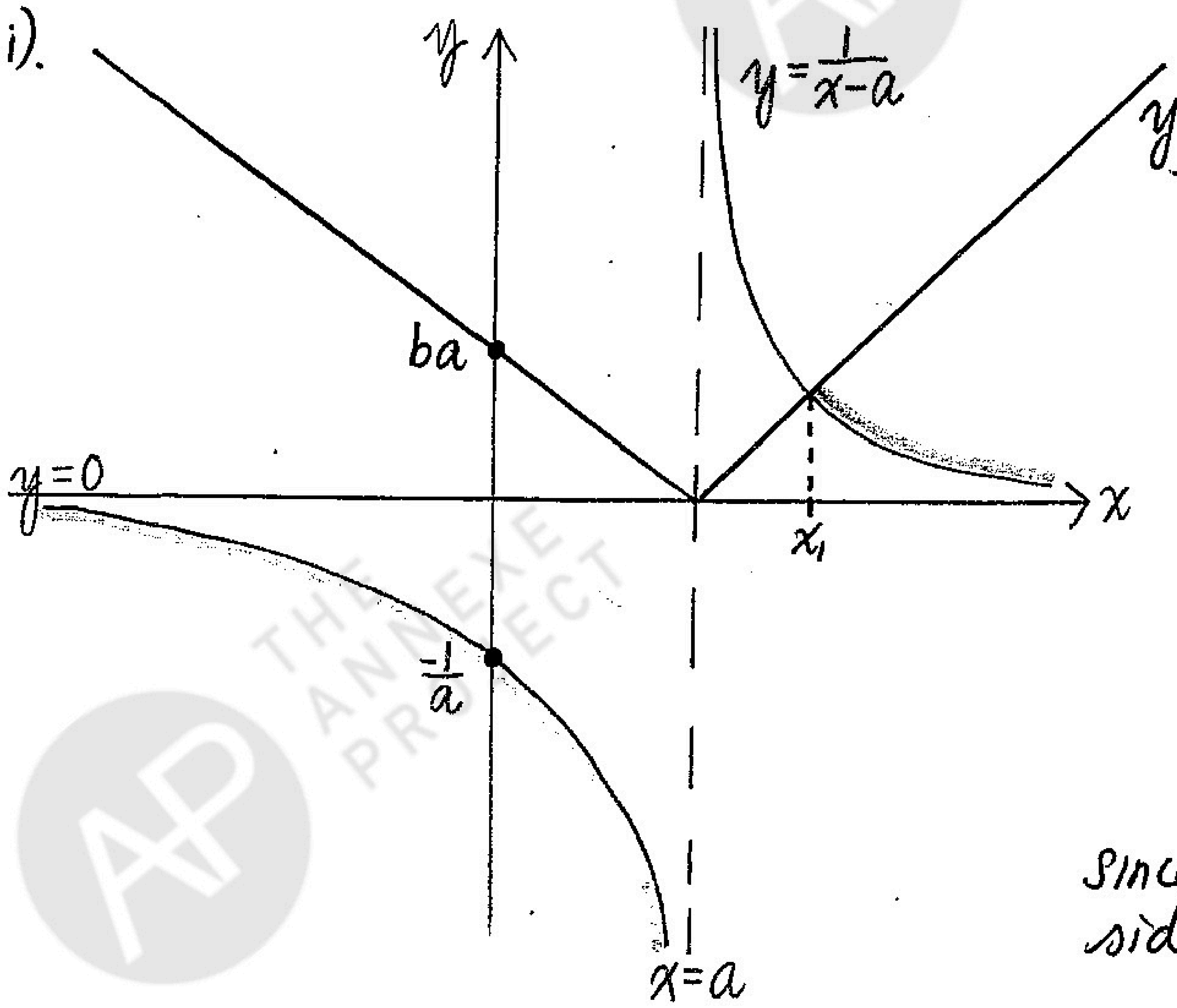
$\underline{a=0 \quad \text{or} \quad a=4}$

(Rej.)

2 (i) On the same axes, sketch the graphs of $y=\dfrac{1}{x-a}$ and $y=b|x-a|$, where a and b are positive constants. [2]

(ii) Hence, or otherwise, solve the inequality $\dfrac{1}{x-a}<b|x-a|$. [4]

i).



ii). Find the intersection between $y=b|x-a|$ and $y=\dfrac{1}{x-a}$:

let $b(x-a)=\dfrac{1}{(x-a)}$

$(x-a)^2=\dfrac{1}{b}$

$x-a=\pm\dfrac{1}{\sqrt{b}}$

$x=a\pm\dfrac{1}{\sqrt{b}}$

Since x_1 is on the right side of $x=a$,

$x_1=a+\dfrac{1}{\sqrt{b}}$

Hence for $\dfrac{1}{x-a}<b|x-a|$,

$\underline{x<a \quad \text{or} \quad x>a+\dfrac{1}{\sqrt{b}}}$

3 **Do not use a calculator in answering this question.**

A curve C has equation $y^2 - 2xy + 5x^2 - 10 = 0$.

(i) Find the exact x-coordinates of the stationary points of C. [4]

(ii) For the stationary point with $x > 0$, determine whether it is a maximum or minimum. [3]

i). Implicit differentiation
differentiate w.r.t x:

$$2y\frac{dy}{dx} - 2x\frac{dy}{dx} + y(-2) + 10x = 0 \quad \text{——— (1)}$$

$$\frac{dy}{dx} = \frac{2y - 10x}{(2y - 2x)}$$

Let $\frac{dy}{dx} = 0$ $\therefore$ $2y - 10x = 0$
$y = 5x$

Sub $y = 5x$ into the equation of the curve:

$$(5x)^2 - 2x(5x) + 5x^2 - 10 = 0$$
$$25x^2 - 10x^2 + 5x^2 = 10$$
$$20x^2 = 10$$
$$x^2 = \frac{1}{2}$$
$$x = \frac{-1}{\sqrt{2}} \text{ or } \frac{1}{\sqrt{2}}$$

ii). For $x = \frac{1}{\sqrt{2}}$;

$$y = 5\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$$

differentiate (1) w.r.t x:

$$2y\frac{d^2y}{dx^2} + \frac{dy}{dx}2\left(\frac{dy}{dx}\right) - 2x\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2) - 2\frac{dy}{dx} + 10 = 0$$

$$(2y - 2x)\frac{d^2y}{dx^2} = 4\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 - 10$$

$$\frac{d^2y}{dx^2} = \frac{4\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 - 10}{2y - 2x}$$

$$= \frac{-10}{2(y - x)}$$

$$= \frac{-10}{2\left[\frac{5}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]} < 0$$

Since $\frac{d^2y}{dx^2} < 0$, $\left(\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$ is a maximum point.

4 A curve C has equation $y = \frac{4x+9}{x+2}$.

(i) Show that the gradient of C is negative for all points on C. [3]

(ii) By expressing the equation of C in the form $y = a + \frac{b}{x+2}$, where a and b are constants, write down the equations of the asymptotes of C. [3]

(iii) Describe a pair of transformations which transforms the graph of C on to the graph of $y = \frac{1}{x}$. [2]

i) $\frac{dy}{dx} = \frac{(x+2)(4) - (4x+9)(1)}{(x+2)^2}$

$= \frac{4x+8-4x-9}{(x+2)^2}$

$= \frac{-1}{(x+2)^2}$ Since $(x+2)^2$ is always positive, $\frac{dy}{dx}$ is always negative for all $x \in \mathbb{R} \setminus \{-2\}$

ii).

$$\begin{array}{r} 4 \\ x+2 \overline{)\ 4x+9} \\ 4x+8 \\ \hline 1 \end{array}$$

$C: y = 4 + \frac{1}{x+2}$

Vertical asymptote: $x = -2$

Horizontal asymptote: $y = 4$

iii).

$y = 4 + \frac{1}{x+2} \longrightarrow y = \frac{1}{x+2} \xrightarrow[\text{with } x-2]{\text{Replace } x} y = \frac{1}{x}$

Step I: translation of 4 units in the negative y-direction

II: translation of 2 units in the positive x-direction

5 When the polynomial $x^3 + ax^2 + bx + c$ is divided by $(x-1)$, $(x-2)$ and $(x-3)$, the remainders are 8, 12 and 25 respectively.

(i) Find the values of a, b and c. [4]

A curve has equation $y = f(x)$, where $f(x) = x^3 + ax^2 + bx + c$, with the values of a, b and c found in part (i).

(ii) Show that the gradient of the curve is always positive. Hence explain why the equation $f(x) = 0$ has only one real root and find this root. [3]

(iii) Find the x-coordinates of the points where the tangent to the curve is parallel to the line $y = 2x - 3$. [3]

i). Let $f(x) = x^3 + ax^2 + bx + c$

$f(1) = 1 + a + b + c = 8$

$a + b + c = 7$ ——— (1)

$f(2) = 8 + 4a + 2b + c = 12$

$4a + 2b + c = 4$ ——— (2)

$f(3) = 27 + 9a + 3b + c = 25$

$9a + 3b + c = -2$ ——— (3)

Solving the 3 eqns simultaneously:

$a = -\frac{3}{2}$, $b = \frac{3}{2}$, $c = 7$

ii). $y = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x + 7$

$\frac{dy}{dx} = 3x^2 - 3x + \frac{3}{2}$

$= 3[x^2 - x + \frac{1}{2}] = 3[(x - \frac{1}{2})^2 + \frac{1}{2} - (\frac{1}{2})^2]$

$= 3[(x - \frac{1}{2})^2 + \frac{1}{4}]$

$= 3(x - \frac{1}{2})^2 + \frac{3}{4}$

> 0 for all $x \in \mathbb{R}$

If this graph's gradient is always positive, then $f(x)$ is a strictly increasing function, i.e it will cut the x-axis only once $\Rightarrow$ hence, one root only.

By GC: $x = -1.3299$

$= -1.33$

iii). Let $\frac{dy}{dx} = 2$

$\therefore 3x^2 - 3x + \frac{3}{2} = 2$

$3x^2 - 3x - \frac{1}{2} = 0$

$6x^2 - 6x - 1 = 0$

by GC: $x = -0.145$ (3 s.f)

or $x = 1.15$ (3 s.f)

6 (i) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter. [2]

(ii) Interpret geometrically the vector equation $\mathbf{r}.\mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents. [3]

(iii) Given that $\mathbf{b}.\mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r}.\mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically. [3]

i). The vector equation $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{b}$ represents the equation of a vector line that passes through a fixed point of position vector $\underset{\sim}{a}$ and is parallel to the direction of vector $\underset{\sim}{b}$.

Any variable vector P on this line can be represented as position vector $\vec{OP} = \underset{\sim}{a} + t\underset{\sim}{b}$, $t \in \mathbb{R}$

ii). The vector equation $\underset{\sim}{r} \cdot \underset{\sim}{n} = d$ represents the equation of a vector plane that has a normal vector $\underset{\sim}{n}$.

d is a numerical value that represents the dot product of any position vector of a point on this plane with the normal vector.

Any variable vector P on this plane, when substituted as $\underset{\sim}{r}$ in the above equation will satisfy the equation.

iii). $(\underset{\sim}{a} + t\underset{\sim}{b}) \cdot \underset{\sim}{n} = d$

$\underset{\sim}{a} \cdot \underset{\sim}{n} + t(\underset{\sim}{b} \cdot \underset{\sim}{n}) = d$

$t = \dfrac{d - (\underset{\sim}{a} \cdot \underset{\sim}{n})}{(\underset{\sim}{b} \cdot \underset{\sim}{n})}$

$\therefore \underset{\sim}{r} = \underset{\sim}{a} + \left[\dfrac{d - (\underset{\sim}{a} \cdot \underset{\sim}{n})}{(\underset{\sim}{b} \cdot \underset{\sim}{n})}\right] \underset{\sim}{b}$

The above solution represents all the possible position vectors of points of intersection between the vector line and the vector plane.

7 It is given that $f(x) = \sin 2mx + \sin 2nx$, where m and n are positive integers and $m \neq n$.

(i) Find $\int \sin 2mx \sin 2nx \, dx$. [3]

(ii) Find $\int_0^{\pi} (f(x))^2 \, dx$. [5]

i). $\int \sin 2mx \sin 2nx \;\; dx$

$= -\frac{1}{2}\int -2 \sin 2mx \sin 2nx \;\; dx$

$= -\frac{1}{2}\int \cos(2m+2n)x - \cos(2m-2n)x \, dx$

$= \frac{-\frac{1}{2}\sin(2m+2n)x}{(2m+2n)} + \frac{\frac{1}{2}\sin(2m-2n)x}{(2m-2n)} + C$

$= \frac{1}{2}\left[\frac{\sin(2m-2n)x}{(2m-2n)} - \frac{\sin(2m+2n)x}{(2m+2n)}\right] + C$

Let $2mx = \frac{1}{2}(P+Q)$

$4mx = P + Q$ —(1)

Let $2nx = \frac{1}{2}(P-Q)$

$4nx = P - Q$ —(2)

$(1)+(2)$: $2P = 4mx + 4nx$

$P = 2mx + 2nx$

$(1)-(2)$: $2Q = 4mx - 4nx$

$Q = 2mx - 2nx$

ii). $\int_0^{\pi} [f(x)]^2 \, dx = \int_0^{\pi} (\sin 2mx + \sin 2nx)^2 \, dx$

$= \int_0^{\pi} \sin^2 2mx + 2\sin 2mx \sin 2nx + \sin^2 2nx \;\; dx$

$= \int_0^{\pi} \frac{1}{2}(1-\cos 4mx) + 2\sin 2mx \sin 2nx + \frac{1}{2}(1-\cos 4nx) \, dx$

$= \int_0^{\pi} 1 - \frac{1}{2}\cos 4mx - \frac{1}{2}\cos 4nx + 2\sin 2mx \sin 2nx \, dx$

$= \left[x - \frac{\frac{1}{2}\sin 4mx}{4m} - \frac{\frac{1}{2}\sin 4nx}{4n} + \frac{\sin(2m-2n)x}{(2m-2n)} - \frac{\sin(2m+2n)x}{(2m+2n)}\right.$

$= [\pi - 0 - 0 + 0 - 0]$

$= \pi$ units^2

8 **Do not use a calculator in answering this question.**

(a) Find the roots of the equation $z^2(1-i)-2z+(5+5i)=0$, giving your answers in cartesian form $a+ib$. [3]

(b) (i) Given that $\omega = 1-i$, find ω^2, ω^3 and ω^4 in cartesian form. Given also that

$$\omega^4 + p\omega^3 + 39\omega^2 + q\omega + 58 = 0,$$

where p and q are real, find p and q. [4]

(ii) Using the values of p and q in part **(b)(i)**, express $\omega^4 + p\omega^3 + 39\omega^2 + q\omega + 58$ as the product of two quadratic factors. [3]

(a). $z = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1-i)(5+5i)}}{2(1-i)}$

$= \dfrac{2 \pm \sqrt{4 - 4(5+5)}}{2(1-i)}$

$= \dfrac{2 \pm \sqrt{-36}}{2(1-i)} = \dfrac{2 \pm 6i}{2(1-i)} = \dfrac{2+6i}{2-2i}$ or $\dfrac{2-6i}{2-2i}$

$= \dfrac{(2+6i)(2+2i)}{4+4}$ or $\dfrac{(2-6i)(2+2i)}{4+4}$

$= \dfrac{4+16i-12}{8}$ $= \dfrac{4-8i+12}{8}$

$= \underline{-1+2i}$ $= \underline{2-i}$

(b) i. Using GC:

$w^2 = -2i$, $w^3 = -2-2i$, $w^4 = -4$

Given $w^4 + pw^3 + 39w^2 + qw + 58 = 0$

$-4 + p(-2-2i) + 39(-2i) + q(1-i) + 58 = 0$

$(54 - 2p + q) + i(-2p - q - 78) = 0 + 0i$

Comparing coefficients:

$q = 2p - 54$ ——— (1)

$q + 2p + 78 = 0$ ——— (2)

Sub (1) into (2):

$2p - 54 + 2p + 78 = 0$

$4p = -24$

$\underline{p = -6}$

$\therefore \underline{q = 2(-6) - 54 = -66}$

ii. Let $w^4 - 6w^3 + 39w^2 - 66w + 58 = 0$

Since all coefficients of the polynomial are real, conjugate complex roots exist.

$$(w-(1-i))(w-(1+i)) = w^2 - w(1+i) - w(1-i) + (1-i)(1+i)$$
$$= w^2 - 2w + 2$$

$$\begin{array}{r|l}
 & w^2 - 4w + 29 \\
\hline
w^2-2w+2 & w^4 - 6w^3 + 39w^2 - 66w + 58 \\
 & w^4 - 2w^3 + 2w^2 \\
\hline
 & -4w^3 + 37w^2 - 66w + 58 \\
 & -4w^3 + 8w^2 - 8w \\
\hline
 & 29w^2 - 58w + 58 \\
 & 29w^2 - 58w + 58 \\
\hline
 & 0
\end{array}$$

Hence, $w^4 - 6w^3 + 39w^2 - 66w + 58$

$$= \underline{(w^2 - 4w + 29)(w^2 - 2w + 2)}$$

9 **(a)** A sequence of numbers $u_1, u_2, u_3, \ldots$ has a sum S_n where $S_n = \sum_{r=1}^{n} u_r$. It is given that $S_n = An^2 + Bn$, where A and B are non-zero constants.

(i) Find an expression for u_n in terms of A, B and n. Simplify your answer. [3]

(ii) It is also given that the tenth term is 48 and the seventeenth term is 90. Find A and B. [2]

(b) Show that $r^2(r+1)^2 - (r-1)^2 r^2 = kr^3$, where k is a constant to be determined. Use this result to find a simplified expression for $\sum_{r=1}^{n} r^3$. [4]

(c) D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, and diverges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$. When $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=0}^{\infty} \frac{x^r}{r!}$ converges for all real values of x and state the sum to infinity of this series, in terms of x. [4]

(a)i. $u_n = S_n - S_{n-1}$

$= (An^2 + Bn) - [A(n-1)^2 + B(n-1)]$

$= \cancel{An^2} + \cancel{Bn} - [\cancel{An^2} - 2An + A + \cancel{Bn} - B]$

$= 2An - A + B$

ii. $u_{10} = 20A - A + B = 48$

$B = 48 - 19A$ ——— (1)

$u_{17} = 34A - A + B = 90$

$B = 90 - 33A$ ——— (2)

$48 - 19A = 90 - 33A$

$14A = 42$

$A = 3$

$\therefore B = 90 - 33(3)$

$= -9$

(b). $LHS = r^2(r+1)^2 - (r-1)^2 r^2$

$= r^2(r^2 + 2r + 1) - (r^2 - 2r + 1)r^2$

$= \cancel{r^4} + 2r^3 + \cancel{r^2} - \cancel{r^4} + 2r^3 - \cancel{r^2}$

$= 4r^3$

$\therefore k = 4$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}\sum_{r=1}^{n} 4r^3$$

$$= \frac{1}{4}\sum_{r=1}^{n} r^2(r+1)^2 - (r-1)^2 r^2$$

$$= \frac{1}{4}\Big[(2^2 - 0)$$

$$+ (2^2\cdot 3^2 - 2^2)$$

$$+ (3^2\cdot 4^2 - 2^2\cdot 3^2)$$

$$+ (4^2\cdot 5^2 - 3^2\cdot 4^2)$$

$$+ \vdots$$

$$+ (n^2\cdot(n+1)^2 - (n-1)^2 n^2) \Big]$$

$$= \underline{\frac{1}{4} n^2 (n+1)^2}$$

(c). Let $a_r = \dfrac{x^r}{r!}$

then $a_{n+1} = \dfrac{x^{n+1}}{(n+1)!}$ and $a_n = \dfrac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n} \times \frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left| x \cdot \frac{1}{(n+1)}\right|$$

$$= 0 \quad (\because \text{ as } n\to\infty,\ \frac{1}{n+1}\to 0)$$

Since $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 0$, the series $\sum_{r=0}^{\infty}\dfrac{x^r}{r!}$ converges for all real values of x. (shown).

$$S_\infty = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

$$= \underline{e^x \text{ (by Standard Series)}}.$$

10 Electrical engineers are installing electricity cables on a building site. Points (x, y, z) are defined relative to a main switching site at $(0, 0, 0)$, where units are metres. Cables are laid in straight lines and the widths of cables can be neglected.

An existing cable C starts at the main switching site and goes in the direction $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. A new cable is installed which passes through points $P(1, 2, -1)$ and $Q(5, 7, a)$.

(i) Find the value of a for which C and the new cable will meet. [4]

To ensure that the cables do not meet, the engineers use $a = -3$. The engineers wish to connect each of the points P and Q to a point R on C.

(ii) The engineers wish to reduce the length of cable required and believe in order to do this that angle PRQ should be $90°$. Show that this is not possible. [4]

(iii) The engineers discover that the ground between P and R is difficult to drill through and now decide to make the length of PR as small as possible. Find the coordinates of R in this case and the exact minimum length. [5]

i). $\overrightarrow{PQ} = \begin{pmatrix} 5 \\ 7 \\ a \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix}$

$l_{PQ} : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ a+1 \end{pmatrix}, \lambda \in \mathbb{R}$

$l_{OC} : \underset{\sim}{r} = \mu \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \mu \in \mathbb{R}$

Let $\begin{pmatrix} 1+4\lambda \\ 2+5\lambda \\ -1+a\lambda+\lambda \end{pmatrix} = \begin{pmatrix} 3\mu \\ \mu \\ -2\mu \end{pmatrix}$

$3\mu = 4\lambda + 1$ ——— (1)

$\mu = 5\lambda + 2$ ——— (2)

Sub (2) into (1):

$3(5\lambda+2) = 4\lambda + 1$

$11\lambda = -5$

$\lambda = \frac{-5}{11}$

$\therefore \mu = 5\left(\frac{-5}{11}\right) + 2$

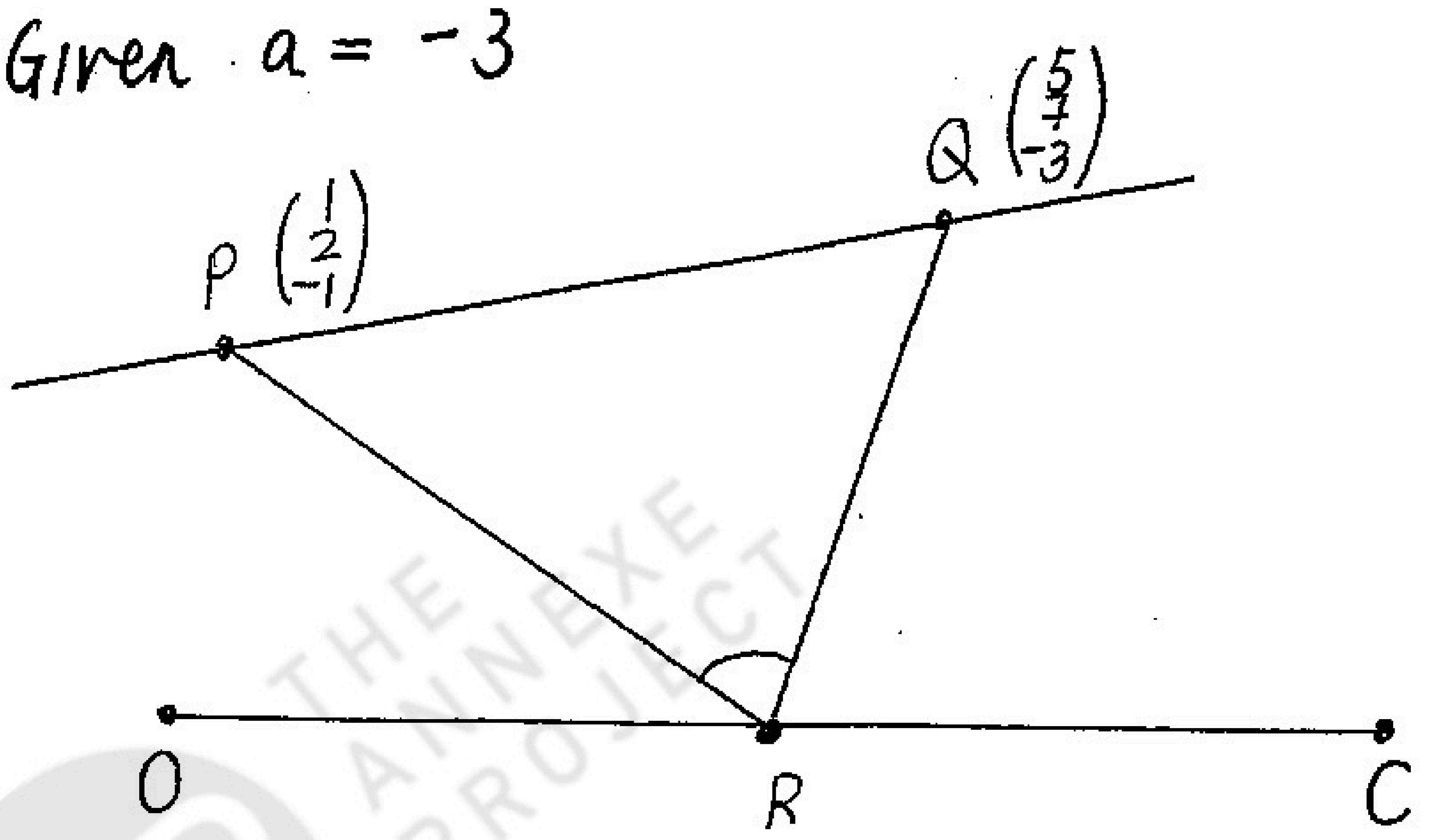
$= \frac{-3}{11}$

Since $-1 - \frac{5}{11}a - \frac{5}{11} = -2\left(\frac{-3}{11}\right)$

$\frac{-5}{11}a = 2$

$\underline{a = -\frac{22}{5}}$

i). Given $a = -3$



Since R is a point on $\overrightarrow{OC}$,

$$\overrightarrow{OR} = \mu\begin{pmatrix}3\\1\\-2\end{pmatrix} = \begin{pmatrix}3\mu\\\mu\\-2\mu\end{pmatrix}$$

$$\overrightarrow{PR} = \begin{pmatrix}3\mu\\\mu\\-2\mu\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}3\mu-1\\\mu-2\\-2\mu+1\end{pmatrix} \quad ; \quad \overrightarrow{RQ} = \begin{pmatrix}5\\7\\-3\end{pmatrix} - \begin{pmatrix}3\mu\\\mu\\-2\mu\end{pmatrix} = \begin{pmatrix}5-3\mu\\7-\mu\\-3+2\mu\end{pmatrix}$$

$$\overrightarrow{PR}\cdot\overrightarrow{RQ} = \begin{pmatrix}3\mu-1\\\mu-2\\-2\mu+1\end{pmatrix}\cdot\begin{pmatrix}5-3\mu\\7-\mu\\-3+2\mu\end{pmatrix}$$

$$= (3\mu-1)(5-3\mu) + (\mu-2)(7-\mu) + (1-2\mu)(2\mu-3)$$

$$= -9\mu^2+18\mu-5-\mu^2+9\mu-14-4\mu^2+2\mu-3$$

$$= -14\mu^2+29\mu-22$$

Let $-14\mu^2+29\mu-22=0$

$$b^2-4ac = (29)^2-4(-14)(-22) = -391 < 0$$

Hence, there are no real values of μ that satisfy $-14\mu^2+29\mu-22=0$

i.e. $\overrightarrow{PR}\cdot\overrightarrow{RQ} \neq 0$ for all real values of μ.

iii). Let $|\overrightarrow{PR}| = S$

$$\therefore S = \sqrt{(3\mu-1)^2+(\mu-2)^2+(1-2\mu)^2} = \sqrt{14\mu^2-14\mu+6}$$

$$\frac{dS}{d\mu} = \frac{1}{2}(14\mu^2-14\mu+6)^{-\frac{1}{2}}(28\mu-14) = \frac{14\mu-7}{\sqrt{14\mu^2-14\mu+6}}$$

Let $\frac{dS}{d\mu}=0 \quad \therefore \quad 14\mu-7=0$

$$\mu = \frac{1}{2}$$

11 Sir Isaac Newton was a famous scientist renowned for his work on the laws of motion. One law states that, for an object falling vertically in a vacuum, the rate of change of velocity, $v\,\mathrm{m\,s^{-1}}$, with respect to time, t seconds, is a constant, c.

(i) (a) Write down a differential equation relating v, t and c. [1]

(b) Initially the velocity of the object is $4\,\mathrm{m\,s^{-1}}$ and, after a further 2.5 s, the velocity of the object is $29\,\mathrm{m\,s^{-1}}$. Find v in terms of t and state the value of c. [3]

For an object falling vertically through the atmosphere, the rate of change of velocity is less than that for an object falling in a vacuum. The new rate of change of v is modelled as the difference between the value of c found in part (i)(b) and an amount proportional to the velocity v, with a constant of proportionality k.

(ii) Given that in this case the initial velocity is zero, find v in terms of t and k. [5]

For an object falling through the atmosphere, the 'terminal velocity' is the value approached by the velocity after a long time.

(iii) A falling object has initial velocity zero and terminal velocity $40\,\mathrm{m\,s^{-1}}$. Find how long it takes the object to reach 90% of its terminal velocity. [4]

(i). a. Let v (in $\mathrm{ms^{-1}}$) be velocity w.r.t time (in seconds).

$$\frac{dv}{dt} = c$$

b. $\int dv = c\int dt$

$v = ct + A$

When $t = 0\,s$, $v = 4\,\mathrm{ms^{-1}}$: $4 = A$

$t = 2.5\,s$, $v = 29\,\mathrm{ms^{-1}}$: $29 = 2.5c + 4$

$c = 10$

$\therefore \underline{v = 10t + 4}$

(ii). $\frac{dv}{dt} = 10 - kv$

$\int \frac{1}{10-kv}\,dv = \int dt$

$\frac{-1}{k}\int \frac{-k}{10-kv}\,dv = \int dt$

$-\frac{1}{k}\ln|10-kv| = t + C$

$\ln|10-kv| = -kt - kC$

$|10-kv| = e^{-kt-kC}$

$|10-kv| = e^{-kt}e^{-kC}$

$10-kv = \pm e^{-kC}e^{-kt}$

$10-kv = Ae^{-kt}$ (where $A = \pm e^{-kC}$)

When $t = 0\,s$, $v = 0\,\mathrm{ms^{-1}}$

$\therefore 10 = Ae^{0}$

$A = 10$

$\Rightarrow 10 - kv = 10e^{-kt}$

$kv = 10 - 10e^{-kt}$

$\underline{v = \frac{1}{k}(10 - 10e^{-kt})}$

When $\mu = \frac{1}{2}$, $\vec{OR} = \begin{pmatrix} 3/2 \\ 1/2 \\ -1 \end{pmatrix}$

coord. of $R = (1.5, 0.5, -1)$

μ	$\frac{1}{2}^-$	$\frac{1}{2}$	$\frac{1}{2}^+$
$\frac{dS}{d\mu}$	\	—	/

When $\mu = \frac{1}{2}$, S is a minimum.

$\therefore S_{min} = \sqrt{14(0.5)^2 - 14(0.5) + 6}$

$= \sqrt{\frac{5}{2}}$ metres .

(iii). As $t \to \infty$, $e^{-Kt} \to 0$

$\therefore v \to \frac{1}{K}(10)$

terminal velocity $v = \frac{10}{K}$

Let $40 = \frac{10}{K}$

$\therefore K = \frac{1}{4}$

$\Rightarrow v = 4(10 - 10e^{-\frac{1}{4}t})$

$0.9(40) = 4(10 - 10e^{-0.25t})$

$10 - 10e^{-0.25t} = 9$

$10e^{-0.25t} = 1$

$e^{-0.25t} = 0.1$

$-0.25t = \ln 0.1$

$\therefore \underline{t = 9.21\ s}$